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IX. *Lines of Induction in a Magnetic Field.*

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[PLATES 14–21.]

THE following paper, which is partly experimental and partly mathematical, has arisen from the discovery that two-dimensional cases of magnetic lines of force could apparently be represented by the flow of a viscous liquid.* The original experiments upon which this assumption was made, showed that the stream lines which were obtained by the method in question, gave results very similar to those which had been calculated and plotted for the cases of an elliptical and circular cylinder. In order to ascertain definitely that the stream lines under these circumstances actually gave the exact position and direction of the corresponding magnetic lines of force, a result which, if verified, could be used for many practical investigations—it was necessary to undertake a long research dealing with the various points involved, a research which has proved extremely laborious, extending without intermission over a period of nearly two years.

In the first place it was necessary to devise some method by which a thin sheet of transparent or semi-transparent medium could be obtained of any required thickness, and on which, when placed between two sheets of glass, the required section of the body to be investigated could be formed.

Next it was necessary to determine the laws connecting the thickness of the thin film of liquid with the quantity flowing through it in a given time, so that the relative differences of thickness corresponding to the differences of permeability of the substances in a magnetic field could be ascertained.

Lastly, a mathematical investigation was undertaken of some cases suitably selected so as to afford, when plotted out, as severe a test as possible for ascertaining if the experimental method really determined for any case, accurately, the position and character of the lines of force in a magnetic field.

It may at once be stated that after overcoming in succession a very large number of difficulties, the case selected, viz., that of an ellipse with the major axis parallel

* "Stream-line Motion of a Viscous Film": 'British Association Report' (Section A), Bristol Meeting, 1898.

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to the lines of force in the undisturbed magnetic field, showed unmistakably absolute agreement between the results obtained by calculation and experiment.

On account of the great practical importance of magnetic phenomena, various methods have from time to time been devised for the study of magnetic fields, and by means of such methods the approximate distribution of the lines of magnetic induction in various cases may be investigated. One of the earliest of these methods was that in which the lines are approximately mapped out by means of iron filings. This method, which has been known for a long time (La Hire mentions it as far back as 1717), was very largely used by Faraday in his study of magnetic fields. It must, however, be regarded as a very rough-and-ready method, and does not enable us to trace out the exact shape of the lines with any degree of accuracy. Another method, which when carefully applied is capable of giving good results, is that in which a magnetic needle is moved from point to point of the field, the consecutive positions of the two ends of the needle being marked, and a line drawn through the points so determined. This method gives the shape of the lines very accurately if the length of the needle is small in comparison with the radius of curvature of the lines. But although in a diagram so obtained the relative intensity of the field at various points may be roughly estimated by noting the convergence or divergence of the lines, yet it is impossible to indicate this variation of intensity by the distance apart of consecutive lines, since there is no means of ascertaining how far apart the consecutive lines should be drawn. A third method, which we owe to the genius of Faraday, and which is one of very great practical importance, is that of a search-coil connected to a ballistic galvanometer. In applying this method, the search-coil is either jerked out of a given position in the field, or turned through a small angle, or else the field is suddenly removed, or reversed, while the search-coil remains stationary. This method, however, notwithstanding its great importance as a method for ascertaining the magnetic flux through a given area in the field, can hardly be regarded as a method for delineating lines of induction. The same remark is applicable to two other methods; that in which the field intensity is ascertained by measuring the resistance of a bismuth spiral; and that in which a conductor conveying a known current is placed in the field, and the pull on the conductor is measured.

The experimental method which the above investigation proves to be accurate, is applicable to two-dimensional problems only. In so far as this is the case, its scope may appear restricted. But when we consider the fact that in the practical applications of magnetism, the bulk of the phenomena with which we have to deal are of the two-dimensional order, it will be recognised that it offers a solution of problems whose interest and importance are by no means inconsiderable. The magnetic field in and around the armature of a dynamo or alternator, and that in and around a cylindrical case which is used for purposes of magnetic shielding, are examples of important practical problems in two dimensions.

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One very important feature which characterises the method, is the fact that it enables us to obtain the exact shape of the lines of induction not only in air, but *in the para-magnetic or dia-magnetic material itself.* This cannot be accomplished by any of the older methods. The method has therefore been applied to determine the exact form of the lines of induction for a number of cases of mathematical interest, several of which have been also plotted by calculation. Inasmuch as the laws connecting the relative thicknesses of two liquid films with the corresponding values of the permeability, have by the above investigation been determined, it is possible to investigate any practical case which may be of interest, and some examples are given of the application of the method in cases of interest to electricians.

The mathematical portion of the work which has been necessary, has not—as far as the authors are aware—hitherto been available in any published form, and although, as will be seen from the historical statement, parts of the problem have been dealt with by various writers, it is hoped that the account given will be found of use in the further investigation of this subject.

The scheme of the paper is as follows:—

I. (*a*) The experimental determination of the relation between the thickness of film and the corresponding rate of flow.
   (*b*) Statement of results and comparison with theory.
   (*c*) Description of appliances used in obtaining the stream-line diagrams.

II. (*a*) Account of the test case worked out mathematically and plotted, and its comparison with the actual photograph obtained by experiment.
   (*b*) Examples of various cases of mathematical interest, and also some of practical importance to the electrical engineer.

III. (*a*) Brief history of mathematical investigation of the subject, and
   (*b*) A general method of dealing with elliptic cylinders and con-focal elliptic shells.

Section I.

(*a*) The liquid is contained in the cylinder A, fig. 1, from which it is forced by means of compressed air, the pressure being determined by the pressure gauge B. It then passes through the cock C (fine adjustment being secured by the lever D which travels over a scale) to the thermometer box, E, where its temperature is recorded upon the thermometer F, and from thence to the slide J.

The slide consists of two sheets of plate-glass fixed within the frame P; a well is cut in the lower sheet into which the liquid is introduced, a channel being formed between the plates by using tin-foil as a border. The correct thickness is obtained by screwing down the plates upon the thickness gauges N, N, N, N, placed at the

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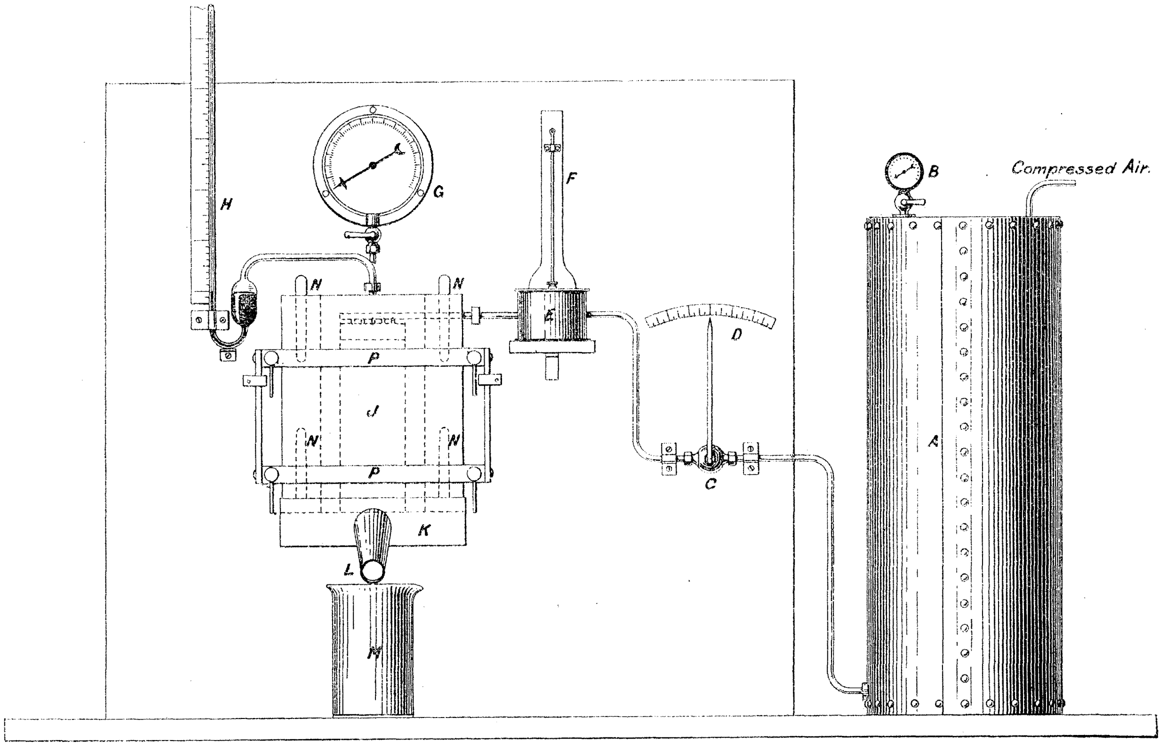
four corners of the slide. The pressure in the well, when low, is found by means of the barometer tube H; when high, by the pressure gauge G.

Fig. 1.



Apparatus for Determining the Viscosity of a Fluid by the method of a Thin Film.

The liquid is caught in the box K from which it passes through the spout L, into the vessel M, where it is collected. When viscous, the liquid was allowed to flow for a fixed time into M, and then weighed; when more mobile, it was collected into a larger vessel of known capacity, and the time observed which it took to fill the vessel up to a notch accurately made for the purpose.

The following tables give the results of the measurements:—

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Experiments on Flow of Thin Film between Parallel Plates.

(1) Glycerine at 20 lbs. pressure.

A = Amount in grammes flowing through in 10 minutes.
B = Time for 1000 grammes to flow.
C = Velocity in centims. per minute.

Thickness.	Temperature.	A.	B.	C.
inch. ·004	° C. 19·2 19·4	19·33 18·66 } 19	minutes. 527	24
·006	17 17 17	60 64 61 } 61·6	162	51·8
·008	17 17 17·2	110 112 108 } 110	91	69·5
·009	17 17 17	170 168 166 } 168	59·5	94·2
·010	16·6 16·6 16·4	195 193 192 } 193·3	51·8	97·5
·011	17 17 17 16·4	271 269 268 267 } 268·75	37·3	125·5
·013	17 16·8 16·8	441 420 428 } 429	23·4	166·5
.014	17 17 16·6	556 559 556 } 557	18	210
·016	17·6 18	862 860 } 861	11·6	271
·017	19·6 19·6 19·5	1090 1076 1070 } 1078·6	9·28	320
·018	17·6	1174 1170 } 1172	8·04	329
·020	17·6 17·6	1640 1658 } 1649	6·05	416

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EXPERIMENTS on Flow of Thin Film between Parallel Plates.

(2) Water at 21 lbs. pressure.

A = Amount in grammes flowing through in 10 minutes.
B = Time for 3850 grammes to flow.
C = Velocity in centims. per minute.

Thickness.	Temperature.	A.	B.	C.
inch. ·004	° C. 16·6 14·2 14·3 13·8	308 307 305·5 305·5 } 306·5	minutes. 126	475
·005	15 15 15·8 15·8 15·8	630 667 665 660 661 } 656	58·7	810
·006	16 16 16	1232 1226 1230 } 1229	31·3	1270
·007	17·5 16·8 15	2050 2058 2040 } 2049	19·2	1817
·008	13·6 13·5 13·5	3040 3048 3040 } 3043·6	12·65	2350
·009	15·2 15·4 15·6	3760	10·28 10·28 10·25 } 10·26	2600
·010	13 12·9 12·8	4470	8·65 8·58 8·58 } 8·6	2760
·011	16·1 16·3 15·2 15 14·6 14·5	6370	6·03 6 6·03 6·03 6·11 6·086 } 6·048	3580
·012	12·4 12·4 12·5 12·4	7050	5·38 5·48 5·5 5·48 } 5·46	3630
·014	14·8 14·8 14·6 14·3	12050	3·166 3·186 3·166 3·23 } 3·187	5325

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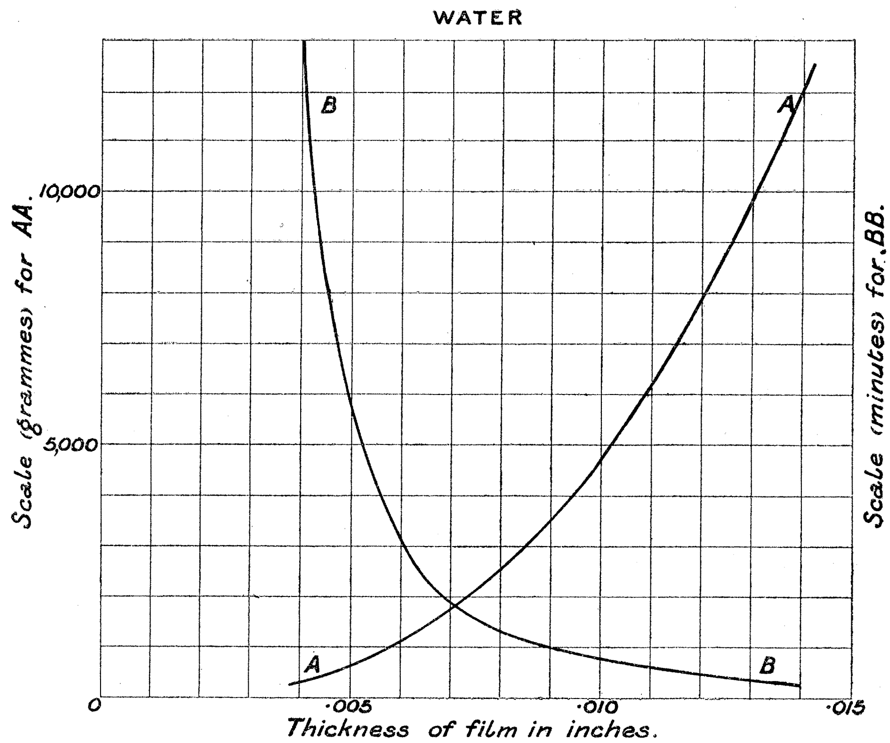
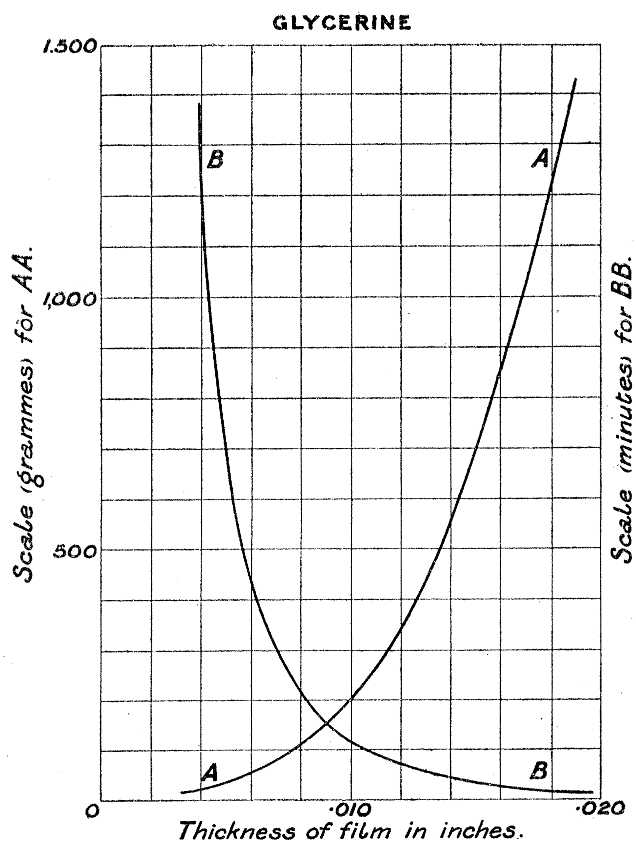
Fig. 2.



AA. Quantity for given time.
BB. Time for given quantity.

Fig. 3.



AA. Quantity for given time.
BB. Time for given quantity.

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From these tables the curves shown in figs. 2 and 3 have been drawn, and these curves were actually used to determine the relation between different depths of well and corresponding permeabilities.

The authors are indebted to Mr. J. C. W. HUMFREY, B.Sc., Victoria University Scholar in Engineering, for valuable assistance in carrying out the above experiments.

(*b*) After the results embodied in the curves, figs. 2 and 3, had been obtained, we thought it would be interesting to see in how far they were in agreement with the deductions from theory. Let

t = thickness of liquid film.
s = tangential stress (per unit area) due to viscosity at a distance x from the middle layer.
v = velocity of flow at distance x from middle layer.
η = coefficient of viscosity.

Then

$$s = \eta \, dv/dx.$$

At a point distant $x = \delta x$ from the middle layer the tangential stress will be

$$s' = \eta \frac{dv}{dx} + \eta \frac{d^2v}{dx^2} \delta x.$$

Hence if we consider a layer of unit length, unit width and thickness δx, the drag to which this elementary layer is subject on account of viscosity is given by

$$\eta \frac{d^2v}{dx^2} \delta x.$$

Now, since we suppose that the flow is steady, the elementary layer considered is not undergoing any acceleration. Hence the backward drag due to viscous resistance must be balanced by the forward push due to the difference of pressure over the two ends of the elementary layer.

If dp/dy stand for the gradient of pressure, then

$$\eta \frac{d^2v}{dx^2} \delta x = - \frac{dp}{dy} \delta x,$$

or

$$\eta \, d^2v/dx^2 = - dp/dy = - f, \text{ say},$$

where f is the fall of pressure per unit length of the liquid layer.

Integrating this equation once, we get

$$\eta \, dv/dx = - fx + C,$$

where C is a constant. Since when $x = 0$, $dv/dx = 0$ (the velocity in the middle being at its maximum), $C = 0$. Integrating a second time, we get

$$\eta v = - \tfrac{1}{2} f x^2 + C'.$$

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In order to find C′, we notice that $v = 0$ when $x = \frac{1}{2}t$, the velocity at the boundary vanishing. This gives

$$v = \frac{f}{2\eta}\left(\frac{t^2}{4} - x^2\right).$$

The volume flowing through per second per unit width of the layer of thickness t is

$$q = \int_{-\frac{1}{2}t}^{+\frac{1}{2}t} v\,dx = 2\int_0^{\frac{1}{2}t} v\,dx$$
$$= \frac{ft^3}{12\eta}.$$

The rate of flow of a viscous liquid in a thin layer between parallel plane walls is thus, for a given fall of pressure along the layer, seen to be proportional to the cube of the thickness of the layer.

An examination of the curves obtained experimentally shows a satisfactory agreement with this theoretical law.

The data contained in the curves also enable us to calculate approximately the coefficient of viscosity. The apparatus is not, however, well adapted for exact measurements, since, in addition to the great difficulty in measuring accurately the thickness of the liquid film, it is doubtful if the glass plates are either sufficiently true or sufficiently rigid (notwithstanding their great thickness) for refined measurements. Since the formula

$$\eta = \frac{ft^3}{12q}$$

for the coefficient of viscosity involves the cube of the thickness of the liquid layer, a comparatively small error in the determination of t will give rise to a large error in η. Taking the water curve (fig. 2), we find for the coefficient of viscosity in C.G.S. units (using the point on the curve corresponding to a thickness of ·012″) the value ·0092, a value which is considerably too low, and which may be accounted for either by an error in estimating the thickness, or by slight irregularities in the containing walls, or slight bulging of the walls in the centre—all of which causes combined might easily account for the error.*

The glycerine curve (fig. 3) gives a value of the viscosity equal to 2·5; the density of the glycerine being 1·23. This value is in fair agreement with the results of previous experimenters.†

As, however, our object was not to carry out careful absolute measurements, but to find relative values which would enable us to apply the method to two-dimensional problems in magnetic induction, and, as the curves obtained (figs. 2 and 3) were

* It must also be remembered that the water used in these experiments was ordinary tap water, which had not been freed from dissolved gases.—July 18, 1900.

† See p. 131 of ARCHBUTT and MOUNTFORD'S 'Lubrication and Lubricants.'

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found to be in agreement with the theoretical law, we did not pursue the subject of viscosity measurements any further.

(*c*) A description of the general method of providing for the flow of colour bands in a thin film has already been given in a paper read before the Institution of Naval Architects.* The apparatus, however, for doing this has been considerably improved and simplified, and shows in figs. 4, 5, and 6, the combination of glass plates which has been devised in connection with the present paper. It will be seen that instead of having three parallel plates of glass, only two are used, in one of which—made very thick—two wells are sunk, used for the clear and coloured liquids respectively. The figures show the construction of the apparatus, and it may be remarked that with this contrivance it is very easy to use a border of paper, cut from a template, and to employ the opposite plate of glass, or the cover-plate, for the purpose of providing the required well or depression through which the liquid flows. The depth of this well is made to correspond to the required magnetic permeability, the outline of the well having the form of a cross-section of the body under investigation. How this well or depression was obtained must now be described, as the difficulties experienced in this case proved much greater than when a solid obstacle was required, as in the paper above mentioned.

In order to solve the problem, it was necessary to obtain a transparent material that could be worked to any shape, and be cemented to the glass cover-plate of the slide to form the recess or well, corresponding in shape to the paramagnetic body for which the lines of induction were required. Originally, for simple circular forms, a sheet of glass such as is used for microscopical work, was cemented to the cover plate, but this was not satisfactory, as the glass could not readily be obtained of the exact thickness required. Further, the thickness of the layer of cement between the plates rendered accuracy very difficult to secure, and any figures other than circles could not easily be cut in glass. Celluloid had the advantage that it could be worked to any shape; but again the difficulty arose that it could not be obtained of the thicknesses required for different resistances, and moreover could not be cemented to the cover plate so as to lie perfectly flat. Various papers and tracing cloths were tried, but they all more or less altered in thickness when moistened, and it was ultimately found that any material requiring to be cemented to the glass could not be used unless it was capable of being brought to a true surface afterwards, as no cement was found capable of holding sufficiently for the object in view. Plates of white metal were thought of, but the labour involved in producing them, and the difficulty of obtaining photographs of the stream-lines, led to the abandonment of this idea also.

The method finally adopted was to cast a layer of paraffin wax on the cover-plate of glass, and although this was not transparent enough to enable the stream-lines

* "Investigation of the Nature of Surface Resistance of Water and of Stream-line Motion under Certain Experimental Conditions." 'Trans. Inst. Naval Architects,' vol. 40, 1898.

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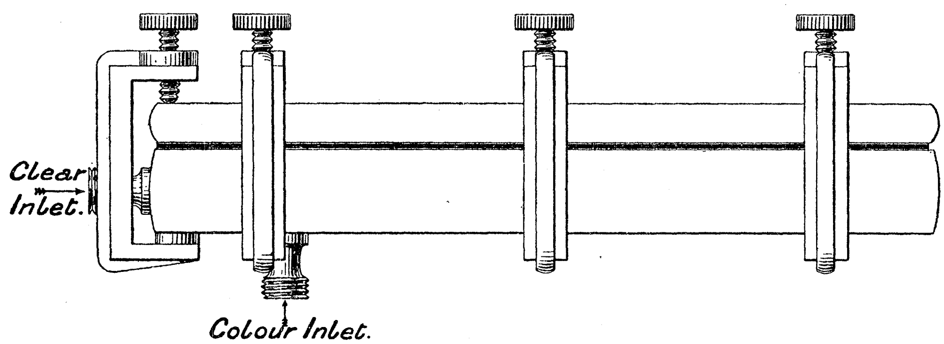
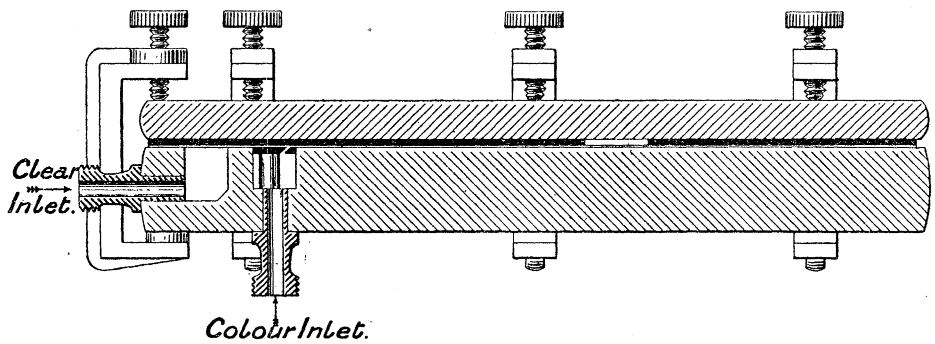
Fig. 4.



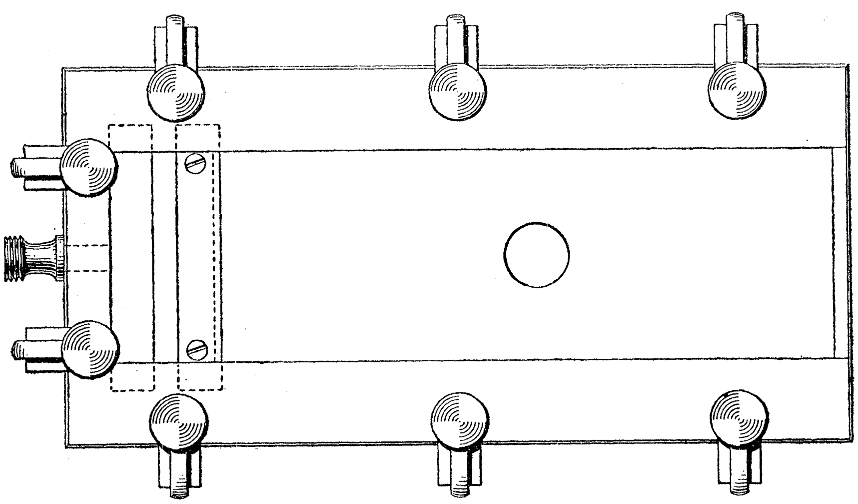
Side view.

Fig. 5.



Sectional view.

Fig. 6.



Front view.

Arrangement of Glass Plates and Clips.

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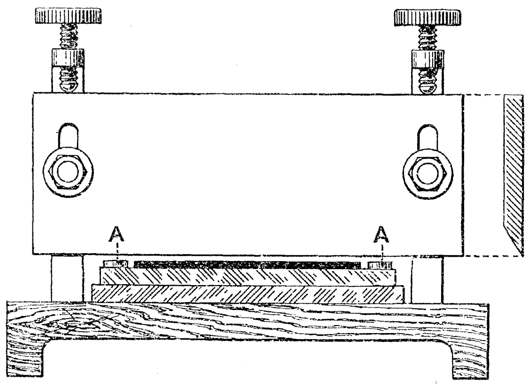
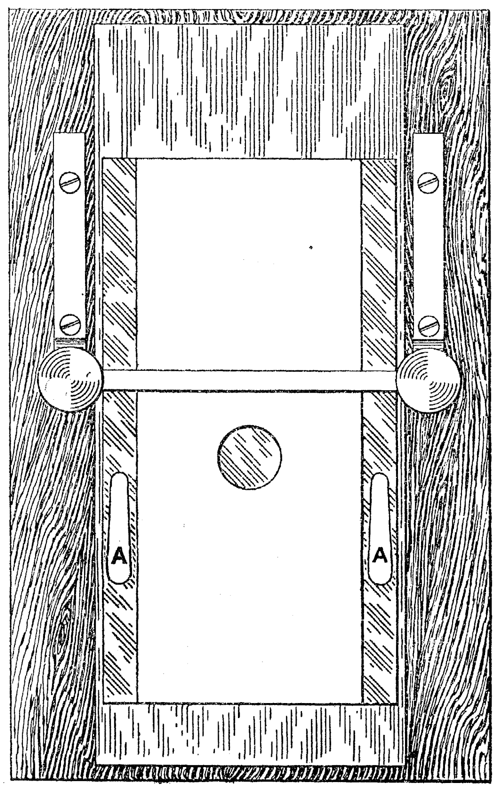
to be projected on the screen, it allowed, when illuminated from behind, sufficient light to pass to enable the stream-lines to be photographed. At first the difficulties of using paraffin wax were numerous, for on casting the wax and allowing it to cool, it became covered with star-like cracks beneath the surface, which were evidently caused by unequal heating and cooling, and imprisoned air bubbles. Casting in a vacuum was also tried, but without success. The difficulty was at length overcome by a systematic method of covering the plates, which was as follows: The plate on which the layer of wax was required, was placed in an iron tray, the wax was shredded on to the plate, and the whole heated gradually and uniformly, a plate of iron 1 inch thick being inserted between the tray and the Bunsen burner used for heating. By this means the plate could be covered to any required thickness in a satisfactory manner. In order to reduce the layer of wax to the thickness required for the experiment, the simple planing machine illustrated in figs. 7 and 8, was used:

Fig. 7.



End elevation.

Fig. 8.



Plan.

Planing Instrument for Paraffin Surface.

the gauges (AA) for obtaining the required thickness being shown in position. The milled-headed screws enabled the cutter to be adjusted so that the desired thickness of wax could be accurately obtained of any dimensions above ·004 inch to within ·001 inch with the greatest ease. The method of illuminating the slide had, however, to be abandoned, as the heat of the arc lamp melted the wax, thus at once putting an end

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to the experiment. Instead of this, the slide was taken to a place where it could be illuminated by strong daylight, and the exposure given to the sensitive photographic plate was of course correspondingly increased. Zinc templates of the shapes required were made, and by placing the template on the wax, the outline was cut with a sharp knife, and the particular well thus formed. In cases where the flow in the well was very slow, air bubbles became imprisoned, and could only be removed after great difficulty ; in fact, in one or two cases—as will be seen in the photographs, figs. 31 and 32 (Plate 19)—they could not be removed at all.

Section II.

(*a*) In order to test the applicability of the stream-line method to the solution of two-dimensional magnetic problems, it was decided to work out mathematically the case of an infinite cylinder of elliptic section placed in an originally uniform magnetic field with its major axis along the field, to plot the lines of magnetic induction corresponding to a permeability of 100, and to compare the diagram so obtained with a stream-line diagram. The theoretical diagram is given in fig. 9 (Plate 14) and the corresponding stream-line diagram in fig. 10. In making the comparison, a greatly enlarged photograph of fig. 10 was prepared, and fig. 9,* which was actually drawn to a much larger scale, was then superposed on it : the coincidence of the lines in the two diagrams satisfactorily established the soundness of the stream-line method. It may be noted, however, that slight local divergences along the elliptic boundary are clearly observable. Instead of the sharp refraction of the lines as they enter the elliptic cylinder in the theoretical diagram, we have in fig. 10 a slight curvature at the ends of the otherwise perfectly straight lines crossing the ellipse. This feature is noticeable, to a greater or smaller extent, in all the stream-line diagrams accompanying the present paper. It is more marked in those cases where the difference between the thicknesses of the two liquid layers, *i.e.*, the permeability in the corresponding magnetic problem, is greater. It is clear that the presence of the highly permeable cylinder disturbs the originally uniform distribution of the lines, and in order to effect a satisfactory comparison between the theoretical and the experimental diagrams, it became necessary to assign to the liquid layer an external boundary whose shape corresponded to that of a particular stream-line in the theoretical diagram. The boundary of the liquid in fig. 10 is clearly shown by the dark shadows on either side of the diagram, and the shape of their boundaries is the same as that of the outer stream-lines in fig. 9. The method by which the solution was obtained for the theoretical case will be found fully explained in the mathematical appendix to the present paper.

(*b*) In most of the succeeding diagrams the boundaries of the liquid layer are straight lines. Interpreted magnetically, this means that the diagram gives the

* For details see Mathematical Section of this paper.

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solution not for a single cylinder of the given cross-section placed in a magnetic field, but of a whole row or grating of such cylinders, the distance apart of any two neighbouring cylinders being equal to the breadth of the liquid film.

Fig. 11 gives the solution for an elliptic cylinder similar to that in fig. 10, but having a permeability of 20. In fig. 12 we again have a cylinder of the same cross-section, but of permeability equal to 1000. On account of the great depth of the elliptic wall, the lines crossing the ellipse appear to be almost entirely obliterated; notwithstanding this fact, they emerge on the other side, preserving their identity and not mixing with the body of the liquid. The distortion of the lines close to and on the inner side of the elliptic boundary is in this case very strongly marked, and well illustrates one of the difficulties encountered in attempting to imitate the effects of highly permeable bodies.

An interesting feature, well-known as a result of theoretical deductions, is clearly brought out by a comparison of the three diagrams, viz., the gradual decrease in the angle made by the external lines with the normal to the ellipse as the permeability is increased. Thus in fig. 11 this angle of incidence of the lines is quite large at certain points of the elliptic boundary; it is greatly reduced in fig. 10, and in fig. 12, for which the permeability is 1000, it is practically zero.

It will be seen that the refraction of the lines in fig. 11 is very sharp—the "weir effect" being extremely feeble. No trouble was experienced on this account in any of the diagrams so long as the permeability did not exceed about 100.

Figs. 13 and 14 (Plate 15) are a set relating to circular cylinders of permeability 2 and 100 respectively. They illustrate clearly the well-known theoretical result that, on account of the shape of the cylinder, large changes of permeability produce only relatively slight changes in the magnetic induction through the cylinder.

In figs. 15—20 we have a set of diagrams which will be found interesting in connection with the important question of magnetic shielding—a subject which has recently attracted a good deal of attention. Fig. 15 is a diagram corresponding to the case of a hollow circular cylindric shield surrounding a solid cylinder, both cylinders having a permeability of 100. Fig. 16 relates to a hollow cylinder whose cross-section is bounded by two confocal ellipses. As shown in the mathematical appendix, the field produced inside such a cylinder is uniform if the original impressed field is uniform. Fig. 17 (Plate 16) represents the effect produced by a double cylindrical shield; it is one of the earliest diagrams obtained by us, and the permeability is so extremely low that, even with the double shield, the field in the innermost space is comparable with the undisturbed field. It would correspond to a highly-saturated double cylindric shield in a field of very great intensity. Figs. 18, 19, and 20 have a more practical interest, the permeability being 100 in each case. In fig. 18 we have a double cylindric shield, with a solid cylinder of iron placed concentrically in the innermost space; the powerful shielding effect on the central cylinder is sufficiently evident. Fig. 20 shows the effect produced by a triple con-

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centric shield. It is interesting to compare this latter stream-line diagram with the theoretical diagram of fig. 19; the agreement in the shape and general distribution of the lines between the two cases is very striking.

Fig. 21 (Plate 17) is a theoretical diagram for an elliptic cylinder of permeability 100, and the ratio of whose axes is 2 : 1, placed in a uniform field with the major axis of the elliptic section of the cylinder making an angle of 45° with the impressed field. Fig. 22 gives a stream-line diagram for a cylinder of the same permeability, but the ratio of whose axes is 3 : 1. Fig. 23 is a similar diagram for a very thin elliptic plate, and fig. 24 relates to the hollow cylinder of fig. 16, but here turned through an angle of 45°. It will be noticed that fig. 24 confirms the theoretical result that the field in the interior of a hollow elliptic cylinder bounded by two confocal surfaces is uniform if the impressed field be uniform.

We have hitherto dealt with cases which may be treated theoretically as well as experimentally. But the number of such cases is very limited, and the vast majority of two-dimensional magnetic problems are beyond the powers of analysis. It is in such cases that the stream-line method employed by us becomes a powerful weapon of research. Figs. 25–28 relate to cylinders of rectangular section, figs. 25 and 26 giving the field distributions for cylinders of square section placed with one of their diagonals at 45° to the field and parallel to it respectively. In fig. 25, where the width of the cylinder remains constant along the direction of the impressed field, we notice that the lines inside the cylinder are concave outwards; in fig. 26, on the other hand, where the cylinder tapers rapidly as we proceed along the field, the curvature of the lines presents a convexity outwards. This suggests that an intermediate form between the two might be found for which the lines exhibit neither convexity nor concavity, *i.e.*, are straight; and, as a matter of fact, we know of one such intermediate form—a circular cylinder. Fig. 26 closely corresponds to the theoretical case which Dr. C. H. LEES has recently succeeded in working out analytically in connection with a problem in heat conduction.* Figs. 27 and 28 are intended to illustrate the effect of increasing the length of one of the sides of the rectangular section while keeping the other constant. We know from theoretical considerations that this has the effect of reducing the de-magnetising factor, and thus increasing the flux through the cylinder. This point is very clearly brought out by a comparison of figs. 25, 27, and 28.

Figs. 29 and 30 (Plate 19) show the magnetic fields corresponding to a cylinder of triangular section in two different positions.

Figs. 31 and 32—a solid circular cylinder inside a hollow square one, and a solid square one inside a hollow circular one—are interesting in connection with the problem of magnetic shielding. Both these diagrams are slightly disfigured by air-bubbles. These latter are extremely difficult to get rid of, once they are allowed to reach a part of the field where the thickness of the liquid film varies.

* 'Phil. Mag.,' February, 1900, p. 225.

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Fig. 33 (Plate 20) is a field diagram for a hollow square cylinder; the shielding effect is seen to be very powerful.

Fig. 35 gives approximately the field distribution between the tapered pole-pieces of an electro-magnet; and fig. 36 is intended to illustrate the pull of an electro-magnet on an armature.

The next few diagrams relate to cases of practical interest and importance, and serve to show how the stream-line method may be made to yield results of great interest to the electrical engineer. Figs. 37 and 38 (Plate 21) show an ordinary SIEMENS shuttle-wound armature—such as is used in connection with telephone call apparatus—in two positions. The direction of the torque acting on the armature in its second position may be at once inferred from an inspection of the diagram. It is also interesting to note the leakage lines outside the armature.

Figs. 34 and 39 relate to a toothed-core armature. Fig. 34 shows the symmetrical field distribution obtained when the teeth are well under cover of the pole-pieces. The induction in the air-gap consists of alternate maxima and minima, and the lines are only slightly curved at the level of the teeth. Below this level, they curve round sharply to enter the flanks of the teeth. The permeability of the core is assumed to be 100.

It is evident that a diagram of this description furnishes a means or calculating the line-integral of magnetic intensity along a line drawn from the polar surface to the base of a tooth, provided the total flux per tooth and the permeability are known.

In fig. 39 we have the same armature, but now shown emerging from under the polar surface. The fringe of the magnetic field is clearly exhibited. The distribution of the lines in the core below the level of the bases of the teeth is not correct. In an actual armature, the lines in this region, after passing into the core, would turn sharply to the left, proceeding towards the neighbouring pole-piece. In the stream-line diagram, on account of the straight-line boundary on the left, the lines are forced to go on in a downward direction. Yet it is curious to note the strong twist towards the left which the lines exhibit immediately after leaving the teeth, and which indicates the direction in which the flow would naturally continue if not subject to the artificial constraint just mentioned.

An objection which might be raised in connection with some of the diagrams is the fact that in those cases where the induction varies from point to point, the permeability in any actual magnetic substance is also variable. In so far as this is the case, of course the stream-line diagrams do not afford a rigid solution of the problem. But it must be remembered that there are many cases in which the magnetic intensity is either so weak or so strong that the permeability does not vary greatly within certain limits of the induction. For all such cases, the solution obtained by the stream-line method is a very close approximation. And even in cases where considerable variations of permeability occur, the stream-line method affords, at any rate, the first rough approximation to the solution of the problem.

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Section III.—*Mathematical Appendix.*

Brief History and General Solution for Elliptic Cylinders and Confocal Elliptic Shells.

(a) The foundation of the modern mathematical theory of magnetic induction was laid by Poisson, between 1821 and 1838. He was the first to work out in detail the solution for a solid or hollow sphere of paramagnetic material placed in a field of uniform intensity.* Subsequently, he extended his investigations to the case of an ellipsoid placed in a uniform field.† Green, at a later date, gave an approximate solution for a cylinder of finite length placed with its axis along a uniform field. In 1848, J. Neumann attacked the problem of an ellipsoid of revolution placed in any given field.‡ In 1854, Kirchhoff succeeded in solving the same problem for a circular cylinder of infinite length. In 1881, A. G. Greenhill considered the case of a hollow ellipsoid.§

Lord Kelvin was the first to publish, in 1872,‖ diagrams of lines of induction for spheres of paramagnetic and diamagnetic material placed in a uniform field. On account of their frequent reproduction (they figure in almost every text-book on the subject), these diagrams are now very well known. In Maxwell's great treatise are to be found a number of two-dimensional diagrams. These include the following cases:—(1) two circular cylinders rigidly magnetised transversely, and placed with their magnetic axes at right angles to each other (vol. 2, fig. 14); (2) a circular cylinder permanently magnetised in a transverse direction, and placed in a uniform field, so that the direction of magnetisation of the cylinder is coincident with that of the field; (3) a cylinder of diamagnetic material in a uniform field (vol. 2, fig. 15); (4) a permanently magnetised cylinder in a uniform field whose direction is at right angles to the direction of magnetisation of the cylinder (vol. 2, fig. 16); (5) a uniform field disturbed by a current in an infinitely long straight cylindrical conductor normal to the direction of the field (vol. 2, fig. 17).

In 1882, Stefan¶ published a long paper dealing with the induced magnetisation of an infinitely long hollow circular cylinder. He obtains a solution by assuming the magnetic potential V to be of the form $(\mathrm{A}r + \mathrm{B}/r)\cos\phi$, where r is the distance of the point considered from the axis of the cylinder, and ϕ the angle between r and the direction of the impressed field; the constants A and B assuming different values for the regions within the cylinder, in its substance, and outside the cylinder respectively. The function $\mathrm{V} = \mathrm{A}x + \mathrm{B}x/r^2$ is an integral of the equation $\frac{\partial^2\mathrm{V}}{\partial x^2} + \frac{\partial^2\mathrm{V}}{\partial y^2} = 0$, and it is the real part of the function $\mathrm{A}z + \mathrm{B}/z$ of the complex

* 'Maxwell,' vol. 2, p. 59.

† *Ibid.*, p. 66.

‡ 'Crelle,' Bd. 37 (1848).

§ 'Journal de Physique' (1881).

‖ 'Reprint of Papers on Electrostatics and Magnetism,' pp. 493–495.

¶ 'Wien. Ber.,' 85, Part 2, p. 613.

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variable $z = x + iy$. The real factor of the imaginary part of this function is $U = Ay - By/r^2$, and this gives the system of lines of induction.

STEFAN further deals with the case of a hollow cylinder along whose axis are placed two wires (infinitely close together), conveying currents in opposite directions.

In 1894, RÜCKER* considered in detail the question of magnetic shielding by spherical shells, with special reference to the problem of maximum shielding effect for a given total weight of material. In the same year PERRY† contributed a brief paper to the Physical Society on the magnetic shielding of external space by a hollow iron cylinder enclosing two parallel conductors conveying equal currents in opposite directions in a diametrical plane of the cylinder, each conductor being at the same distance from the axis.

In 1897, fresh interest was given to the problem of magnetic shielding by the discussion which followed the reading of Mr. MORDEY'S paper on "Dynamos," before the Institution of Electrical Engineers.‡ Towards the close of that year, DU BOIS§ commenced the publication of an elaborate series of articles on magnetic shielding, in which he briefly reviewed the history of the subject, and gave diagrams of magnetic fields for the case of hollow circular cylinders of varying thickness placed in a uniform field.‖

Almost simultaneously with DU BOIS, SEARLE¶ published, in January 1898, a mathematical paper on the magnetic field due to a current in a wire placed parallel to the axis of a cylinder of iron. This paper is accompanied by some extremely interesting diagrams of magnetic fields.

The latest contribution to this subject is a paper by A. P. WILLS,** "On the Magnetic Shielding Effect of Trilamellar Spherical and Cylindrical Shells," which may be considered as an extension of DU BOIS' investigations to triple shields.

(*b*) The problem of the induced magnetisation due to a uniform impressed field in infinite cylinders of elliptic section or infinite cylindrical shells bounded by confocal elliptic surfaces, may be dealt with by the following method.

The general problem of magnetic induction in space of two dimensions may be regarded as consisting in the determination of a continuous potential function V, *i.e.*, a function satisfying LAPLACE'S equation for two-dimensional space, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, which fulfils the condition of continuity of normal induction across any surface of separation between two media, viz.,

$$\mu_1 \frac{\partial V}{\partial n_1} + \mu_2 \frac{\partial V}{\partial n_2} = 0,$$

* 'Phil. Mag.' [5], vol. 37, p. 95.

† 'Proc. Phys. Soc.,' vol. 13, p. 227.

‡ 'Journal,' vol. 26, p. 564.

§ 'The Electrician,' vol. 40.

‖ *Ibid.*, vol. 40, p. 513.

¶ *Ibid.*, vol. 40, p. 453.

** 'Physical Review' [9], pp. 193–213, October, 1899

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where μ_1 and μ_2 stand for the permeabilities of the two media, and n_1, n_2 for the normals drawn from any point of the surface *into* the corresponding media.

In dealing with elliptic cylinders and cylindrical shells, it is convenient to abandon the use of Cartesian rectangular co-ordinates, and to have recourse to the circular and hyperbolic functions.

We shall take

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots \ldots \ldots \quad (1)$$

as our ellipse of reference. Then

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

where λ is a variable parameter such that $\lambda > - b^2$, represents the family of ellipses confocal with (1), while

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} = 1,$$

where ν is a variable parameter such that $- b^2 > \nu > - a^2$, represents the family of hyperbolas confocal with (1).

If now we put

$$\frac{x}{\sqrt{a^2 + \lambda}} = \cos \theta, \quad \frac{y}{\sqrt{b^2 + \lambda}} = \sin \theta,$$

$$\frac{x}{\sqrt{a^2 + \nu}} = \cosh u, \quad \frac{y}{\sqrt{-(b^2 + \nu)}} = \sinh u,$$

then $u =$ constant is the equation to a certain ellipse, and $\theta =$ constant to a hyperbola, both curves being confocal with (1).

It is obvious that u (which may vary from 0 to ∞), and θ (which may vary from 0 to 2π), completely and uniquely determine the position of a point in the plane of the ellipse (1). We shall use u and θ as our co-ordinates.

In order to arrive at the form of the induced potential function, we make use of the following considerations. Poisson has shown that the magnetisation of a solid ellipsoid placed in a uniform field is uniform. Let one of the axes of the ellipsoid be made infinite. Then we pass from the three-dimensional case of the ellipsoid to the two-dimensional one of the elliptic cylinder, and we see that in this case also the magnetisation is uniform. Such a magnetisation might be supposed to be produced by imagining two solid cylinders of the imaginary magnetic matter, of volume-density ρ and of opposite sign, originally coincident, to be displaced relatively to each other through a small distance δs in the direction of magnetisation, such that $\rho \delta s$ is equal to the actual intensity of magnetisation of the cylinder. If then ρV stand for the potential at any point due to one of the solid cylinders of the imaginary magnetic matter, the combined potential of the pair of displaced cylinders is $\rho \frac{\partial V}{\partial s} \delta s$. But

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$\rho \frac{\partial V}{\partial s}$ is the intensity-component in the direction of δs due to one of the cylinders. Thus the problem of finding the magnetic potential function is reduced to that of determining the intensity due to a solid cylinder of attracting or repelling matter. The components of intensity due to this latter cylinder may be shown to be

$$X = \frac{4\pi ab\rho}{\sqrt{a^2 - b^2}} e^{-u} \cos \theta,$$

$$Y = \frac{4\pi ab\rho}{\sqrt{a^2 - b^2}} e^{-u} \sin \theta.$$

We are thus led to adopt tentatively the somewhat more general forms

$$(A \cosh u - B \sinh u) \cos \theta$$

and

$$(C \cosh u - D \sinh u) \sin \theta,$$

where A, B, C, and D are constants, as the typical magnetic potential functions for magnetised elliptic cylinders and confocal elliptic cylindric shells.

In the above expressions, A, B, C and D will all be different for the space included between two confocal bounding surfaces; $A = B$ and $C = D$ for the external space extending to infinity. And $B = 0$, $C = 0$ for the space inside the innermost bounding surface, which includes the foci.

If we assume the permeability to be constant, then we need only consider the two standard cases of magnetisation along the two axes of the ellipse (1), as any intermediate direction may be obtained by a proper superposition of the two standard cases.

We shall in the first place consider the case of a solid cylinder whose bounding surface is the ellipse of reference (1), the impressed field H being along the major axis of the ellipse.

If V_o, V_i and V_e stand for the impressed potential, the induced potential inside, and that outside the cylinder respectively, then

$$V_o = -Hx = -H\sqrt{a^2 - b^2} \cos \theta \cosh u,$$

and we assume

$$V_i = Ax = A\sqrt{a^2 - b^2} \,.\, \cos \theta \cosh u,$$

$$V_e = B\sqrt{a^2 - b^2} \,.\, e^{-u} \cos \theta.$$

In order that these functions—which satisfy LAPLACE'S equation—may afford a solution of the problem, they must satisfy the conditions of (1) continuity of potential and (2) continuity of normal induction—two conditions which enable us to determine the constants A and B.

The first condition $\left(\text{when } u = \tanh^{-1}\frac{b}{a}\right)$ gives

$$aA - (a - b)B = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

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The second gives

$$\mu \frac{\partial(V_o + V_i)}{\partial n} = \frac{\partial(V_o + V_e)}{\partial n}, \quad \text{or} \quad (\mu - 1)\frac{\partial V_o}{\partial n} = \frac{\partial V_e}{\partial n} - \mu\frac{\partial V_i}{\partial n},$$

where μ is the permeability, and n is the outward-drawn normal at any point of the ellipse (1).

Now $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial u} \cdot \frac{du}{dn}$, so that the last equation becomes

$$(\mu - 1)\frac{\partial V_o}{\partial u} = \frac{\partial V_e}{\partial u} - \mu\frac{\partial V_i}{\partial u}.$$

Carrying out the differentiations, and then putting $u = \tanh^{-1}\frac{b}{a}$ we get for the second condition

$$\mu b A + (a - b) B = (\mu - 1) b H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Solving (2) and (3) for A and B, we find

$$A = \frac{\mu - 1}{\mu b + a} \cdot bH$$

$$B = \frac{\mu - 1}{\mu b + a} \cdot \frac{ab}{a - b} \cdot H.$$

We thus have

$$V_o + V_i = -\frac{a + b}{\mu b + a} \cdot Hx$$

$$V_o + V_e = \frac{\sqrt{a^2 - b^2} \cdot H}{(\mu b + a)(a - b)} \cdot \{(\mu b^2 - a^2)\cosh u \cos\theta - (\mu - 1) ab \sinh u \cos\theta\}$$

and the equipotential lines may at once be plotted from these equations.

The equation to the lines of induction inside the cylinder is obviously $y =$ constant.

In order to find the equation to the external lines, we have to determine the function which is conjugate to $V_o + V_e$. Since the function which is conjugate to $\cosh u \cos\theta$ is $\sinh u \sin\theta$, and that conjugate to $\sinh u \cos\theta$ is $\cosh u \sin\theta$, we have

$$K \operatorname{cosec}\theta = \cosh u - \frac{\mu b^2 - a^2}{(\mu - 1) ab} \sinh u$$

for the equation to the lines of the external field, K being a constant which varies from one line to another.

If we next assume that the direction of the impressed field is along the minor axis of the ellipse, we similarly find for the equation to the lines of the external field

$$K \sec\theta = \cosh u - \frac{(\mu - 1) ab}{\mu a^2 - b^2} \sinh u,$$

K being as before a parameter which varies from one line to another.

The treatment given above for the case of a solid elliptic cylinder may be extended

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to any number of hollow confocal elliptic cylinders of different permeabilities. For if (taking the direction of H to be along the major axis of the ellipse) we assume that the induced potential V_i in the innermost space is of the form

$$V_i = A\sqrt{a^2 - b^2} \,.\, \cosh u \cos\theta,$$

that the induced potential V_n in the substance of any one of the hollow confocal elliptic cylinders is

$$V_n = \sqrt{a^2 - b^2}(A_n \cosh u + B_n \sinh u)\cos\theta,$$

and that finally the induced potential in external space is

$$V_e = A_e\sqrt{a^2 - b^2} \,.\, e^{-u}\cos\theta,$$

then all the functions V_i . . . V_n . . . V_e . . . satisfy LAPLACE's two-dimensional equation, and the last function V_e vanishes at an infinite distance. If, therefore, we can determine the various constants A, . . . A_n, . . . A_e, so as to satisfy (1) continuity of potential, and (2) continuity of normal induction, the only possible solution of the problem will have been obtained.

Let us suppose that there are m hollow cylinders. The number of constants in the assumed expressions for the potential will in that case be $2m + 2$. There are $m + 1$ bounding surfaces, and at every such surface the two conditions of continuity of potential and continuity of normal induction must be fulfilled, thus giving two conditional equations for each boundary. There will therefore be $2(m+1)$ equations, and these will completely determine the values of the $2m + 2$ constants.

By way of further illustration, we shall work out the case of a hollow elliptic cylinder of iron placed in a uniform field. Let the internal bounding surface be the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

or $u = \tanh^{-1}(b/a)$, and the external bounding surface the ellipse

$$\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1,$$

or $$u = u_1, \text{ say, where } u_1 = \tanh^{-1}\sqrt{\frac{b^2 + \lambda_1}{a^2 + \lambda_1}}.$$

Let the impressed potential V_o be

$$V_o = -H\sqrt{a^2 - b^2} \,.\, \cosh u \cos\theta,$$

corresponding to a direction of the field along the major axis of the cylinder.

We assume for the induced magnetic potential functions :—

$$V_i = A\sqrt{a^2 - b^2} \,.\, \cosh u \cos\theta,$$

$$V_1 = A\sqrt{a^2 - b^2}(A_1 \cosh u + B_1 \sinh u)\cos\theta$$

and $$V_e = A_e\sqrt{a^2 - b^2} \,.\, e^{-u}\cos\theta.$$

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When $u = \tanh^{-1}(b/a)$, $V_i = V_1$, and when $u = u_1$, $V_1 = V_e$; this gives

$$Aa = aA_1 + bB_1 \quad \ldots \quad (4)$$

and

$$A_1 \cosh u_1 + B_1 \sinh u_1 = A_e \cosh u_1 - A_e \sinh u_1 \quad \ldots \quad (5).$$

Again, when $u = \tanh^{-1}(b/a)$, we must have

$$(\mu - 1)\frac{\partial V_o}{\partial u} = \frac{\partial V_i}{\partial u} - \mu \frac{\partial V_1}{\partial u};$$

and when $u = u_1$,

$$(\mu - 1)\frac{\delta V_o}{\delta u} = \frac{\delta V_e}{\delta u} - \mu \frac{\delta V_1}{\delta u}.$$

These two conditions give

$$-(\mu - 1)bH = bA - \mu bA_1 - \mu aB_1 \quad \ldots \quad (6),$$

$$-(\mu - 1)H \sinh u_1$$
$$= A_e \sinh u_1 - A_e \cosh u_1 - \mu A_1 \sinh u_1 - \mu B \cosh u_1 \quad \ldots \quad (7)$$

Solving equations (4), (5), (6), and (7) for A, A_1, B_1, and A_e, we get

$$A = \frac{(\mu a - b)(a \sinh u_1 - b \cosh u_1)}{(\mu a^2 - b^2)(\mu \sinh u_1 + \cosh u_1) - (\mu - 1)ab(\mu \cosh u_1 + \sinh u_1)} \cdot (\mu - 1)H,$$

$$A_1 = \frac{(\mu a^2 - b^2)\sinh u_1 - ab(\mu \cosh u_1 + \sinh u_1)}{(\mu a^2 - b^2)(\mu \sinh u_1 + \cosh u_1) - (\mu - 1)ab(\mu \cosh u_1 + \sinh u_1)} \cdot (\mu - 1)H,$$

$$B_1 = \frac{a}{b}(A - A_1),$$

$$A_e = \frac{A_1 \cosh u_1 + B_1 \sinh u_1}{e^{-u_1}}.$$

These expressions may be reduced to a somewhat simpler form by putting $\sqrt{a^2 + \lambda_1} = a_1$, $\sqrt{b^2 + \lambda_1} = b_1$, so that a_1, b_1 stand for the semi-axes of the external elliptic bounding surface. Remembering that $u_1 = \tanh^{-1}(b_1/a_1)$, we get

$$A = \frac{(\mu a - b)(\mu - 1)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot H$$

$$A_1 = \frac{\left\{\mu a - (a + b)\dfrac{bb_1}{ab_1 - a_1 b}\right\}(\mu - 1)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot H$$

$$B_1 = \frac{ab(a_1 + b_1)/(ab_1 - a_1 b)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot (\mu - 1)H$$

$$A_e = \frac{(\mu aa_1 + bb_1)/(a_1 - b_1)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot (\mu - 1)H$$

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The equations to the lines of induction are :—

For the central space inside the shell, y = constant.

For the space occupied by the substance of the shell,

$$K \operatorname{cosec} \theta = \cosh u - \frac{H - A_1}{B_1} \sinh u,$$

and for the external space

$$K \operatorname{cosec} \theta = \cosh u - \frac{A_e - H}{A_e} \sinh u,$$

K being in each case a variable parameter.

If we suppose that the impressed field is along the minor axis of the ellipse, so that

$$V_o = - H \sqrt{a^2 - b^2} \sinh u \sin \theta,$$

then, assuming

$$V_i = A' \sqrt{a^2 - b^2} \,.\, \sinh u \sin \theta,$$

$$V_1 = \sqrt{a^2 - b^2} \,(A'_1 \cosh u + B'_1 \sinh u) \sin \theta,$$

and

$$V_e = \sqrt{a^2 - b^2} \,.\, A'_e e^{-u} \sin \theta,$$

we find, proceeding as before

$$A' = \frac{(\mu b - a)(b \cosh u_1 - a \sinh u_1)}{(\mu b^2 - a^2)(\mu \cosh u_1 + \sinh u_1) - (\mu - 1) ab (\mu \sinh u_1 + \cosh u_1)} \,.\, (\mu - 1) H,$$

$$A_1' = \frac{abe^{u_1}}{(\mu b^2 - a^2)(\mu \cosh u_1 + \sinh u_1) - (\mu - 1) ab (\mu \sinh u_1 + \cosh u_1)} \,.\, (\mu - 1) H,$$

$$B_1' = A' - \frac{a}{b} A_1',$$

$$A_e' = \frac{A_1' \cosh \mu_1 + B_1' \sinh u_1}{e^{-u_1}} \,.$$

Or, in terms of a_1, b_1,

$$\left.\begin{aligned} A' &= \frac{\mu b - a}{\Delta} \,.\, (\mu - 1) H, \\ A_1' &= \frac{ab(a_1 + b_1)/(a_1 b - a b_1)}{\Delta} \,.\, (\mu - 1) H, \\ B_1' &= \frac{\mu b - \dfrac{a a_1 (a + b)}{a_1 b - a b_1}}{\Delta} \,.\, (\mu - 1) H, \\ A_e' &= \frac{(\mu b b_1 + a a_1)/(a_1 - b_1)}{\Delta} \,.\, (\mu - 1) H, \end{aligned}\right\} \text{where } \Delta = \mu^2 b + a - \mu (a + b) \frac{a a_1 - b b_1}{a_1 b - a b_1}.$$

The equations to the lines of induction are :—

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$$x = \text{constant, in the innermost space,}$$

$$K \sec\theta = \sinh u + \frac{B_1' - H}{A_1'} \cosh u, \text{ in the substance of the shell, and}$$

$$K \sec\theta = \sinh u - \frac{A_e' + H}{A_e'} \cosh \mu, \text{ in external space.}$$

In working out numerical examples with the view of plotting curves, it is convenient to use a table of hyperbolic sines and cosines, such as the one compiled by T. H. BLAKESLEY and published by the Physical Society of London. Corresponding values of u and θ having been found, it is an easy step to pass to Cartesian rectangular co-ordinates, which are more convenient for plotting the curves.

The method given above may, as already mentioned, be extended to any number of confocal elliptic cylindric shells. In general, however, when numerical data are available, it is best to substitute these at once in the equations instead of first trying to obtain a solution in general terms.

The case of concentric circular cylindric shells may be regarded as a limiting case of elliptic shells, the two axes of the ellipse becoming equal. If in the expressions obtained for the constants A, A_1, &c., in the case of a hollow elliptic shell we regard a as constant and b as variable, and then proceed to the limit $b = a$, we find that the values so obtained are in agreement with those deduced by DU BOIS for a circular cylindric shell in his articles on "Magnetic Shielding."* But although an interesting verification of the method for a special case is thus obtained, it is much simpler, when dealing with circular cylindric shells, to follow the method developed by STEFAN in the paper already referred to.†

* 'The Electrician,' vol. 40.

† 'Wien. Akad. Sitz. Ber.,' vol. 85, Section 2, p. 613.

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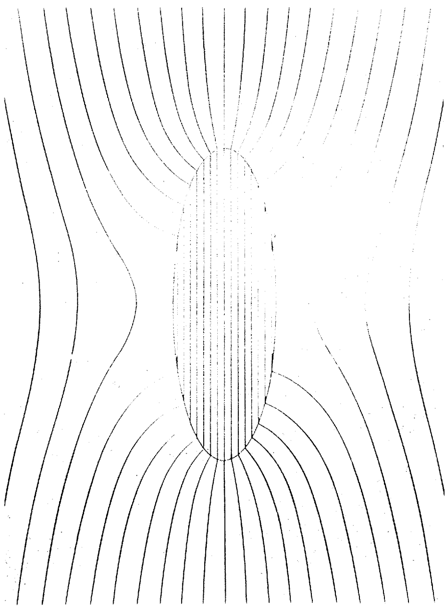
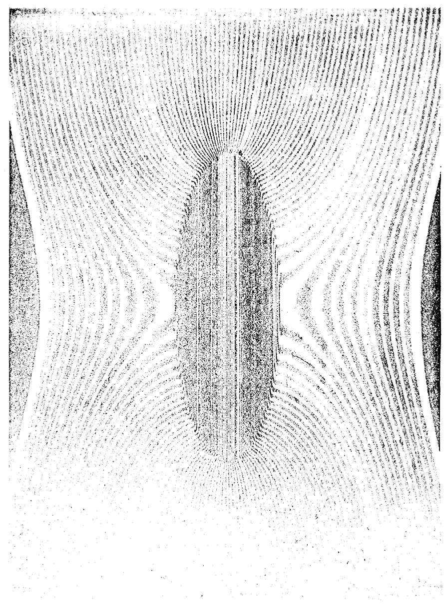


9

Theoretical Diagram for Infinite Elliptic Cylinder placed in uniform field. Ratio of axes 3 : 1. Permeability=100.



10

Stream-line Diagram corresponding to theoretical Diagram of Fig. 9.

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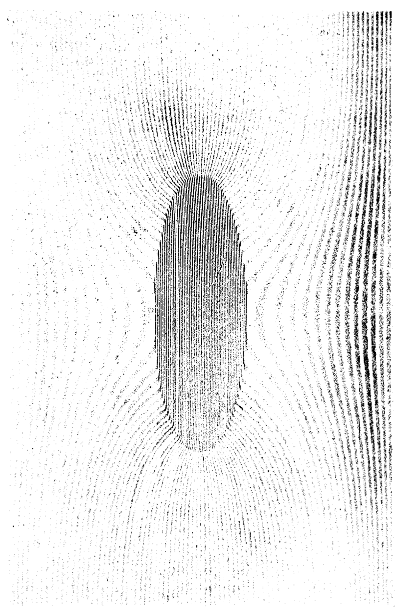
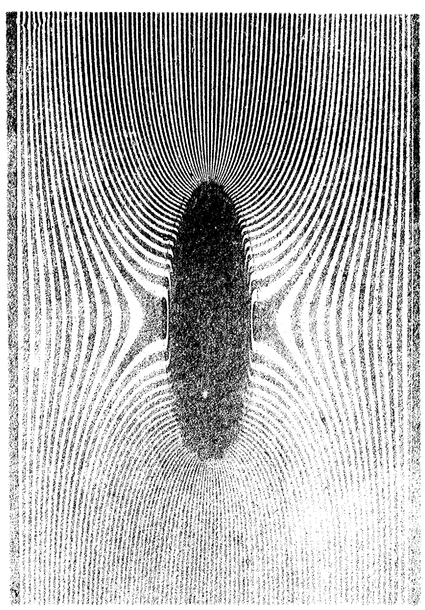


11

Infinite Elliptic Cylinder in uniform field. Ratio of axes 3 : 1. Permeability=20.



12

Infinite Elliptic Cylinder in uniform field. Ratio of axes 3 : 1. Permeability 1000.

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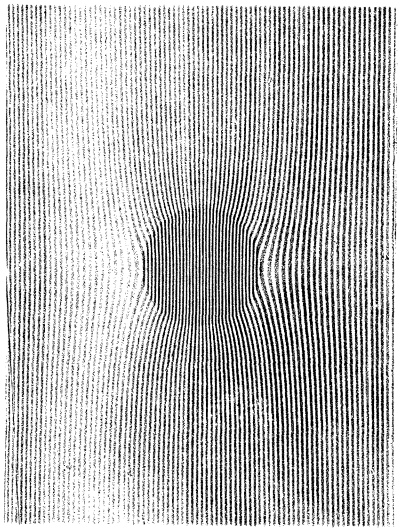

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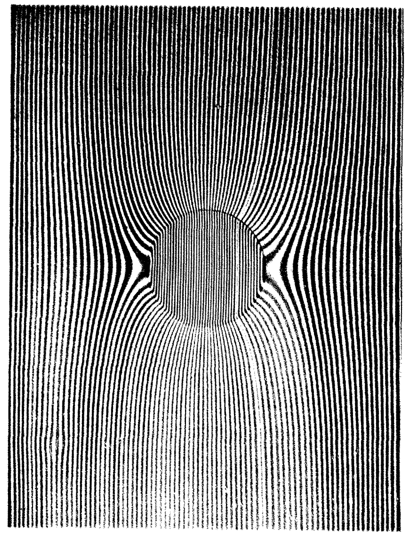
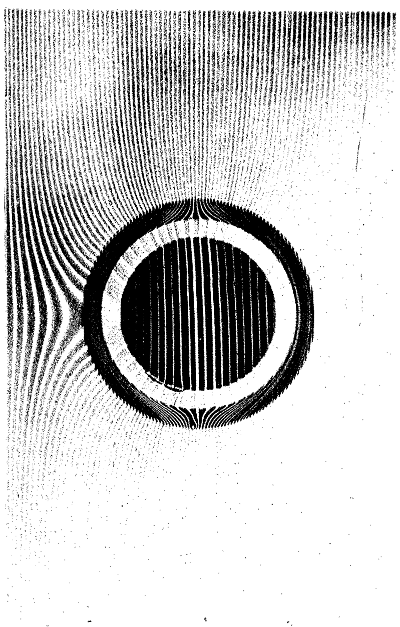
Infinite Circular Cylinder in uniform field. Permeability=2.



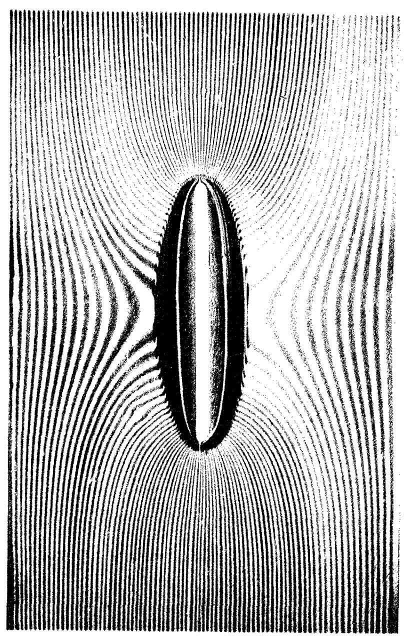
14

Infinite Circular Cylinder in uniform field. Permeability=100.



15

Solid Circular Cylinder shielded by hollow one. Permeability=100.



16

Infinite Hollow Elliptic Cylinder in uniform field. Permeability=100.

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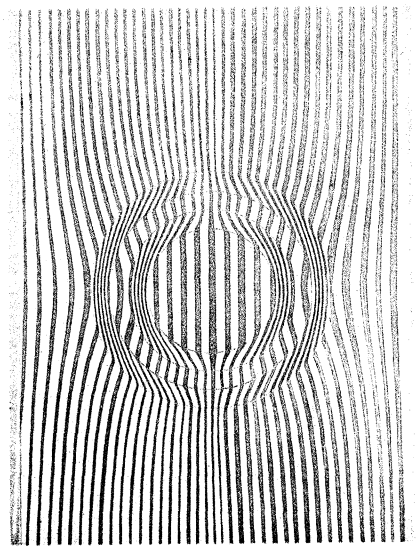
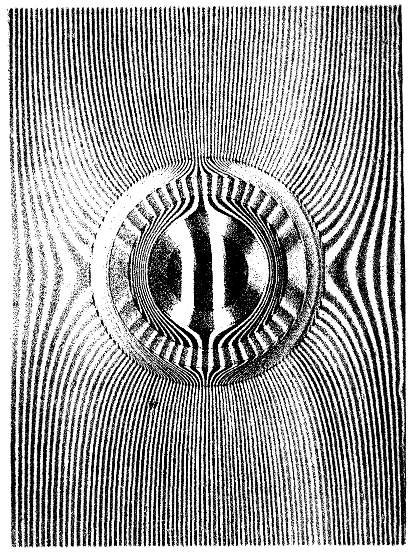


17

Double Cylindric Shield of very low permeability.



18

Double Cylindric Shield enclosing Solid Cylinder. Permeability=100.

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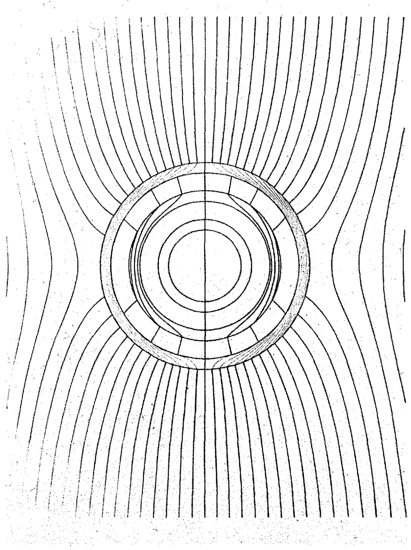
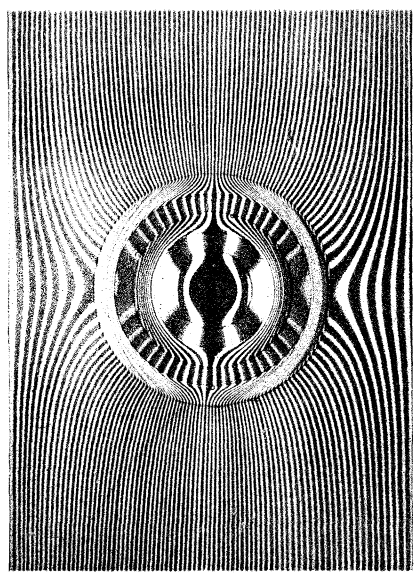


19

Triple Circular Cylindric Shield. Theoretical Diagram. Permeability=100.



20

Triple Circular Cylindric Shield. Stream-line Diagram. Permeability=100.

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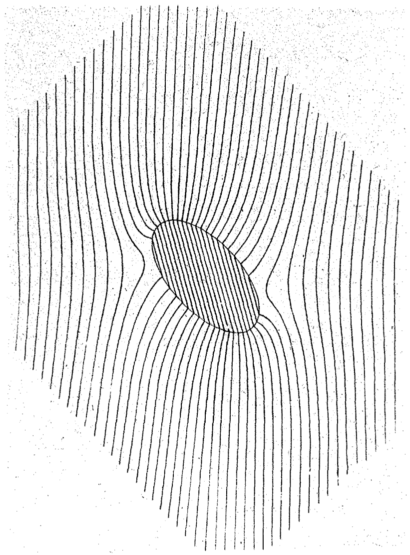
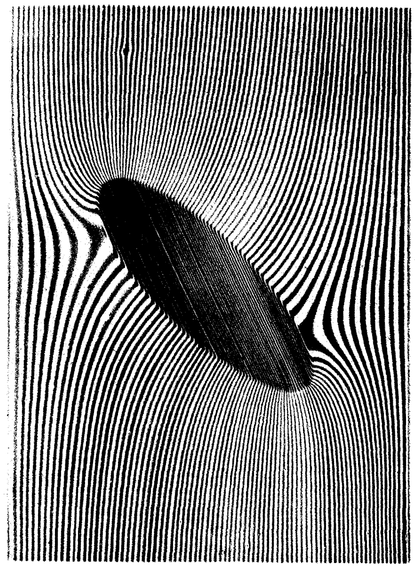


21

Elliptic Cylinder. Ratio of axes 2 : 1, major axis inclined at 45° to field. Theoretical Diagram. Permeability = 100.



22

Elliptic Cylinder. Ratio of axes 3 : 1, major axis inclined at 45° to field. Stream-line Diagram. Permeability = 100.

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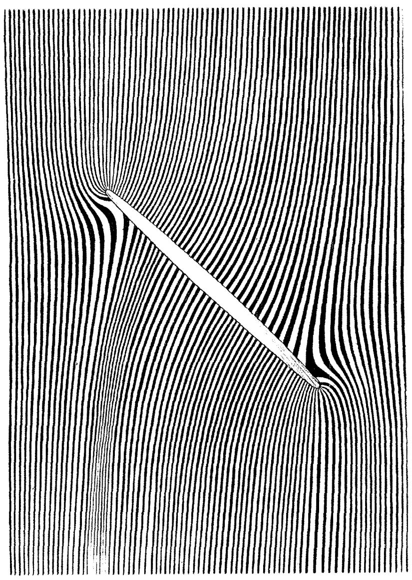
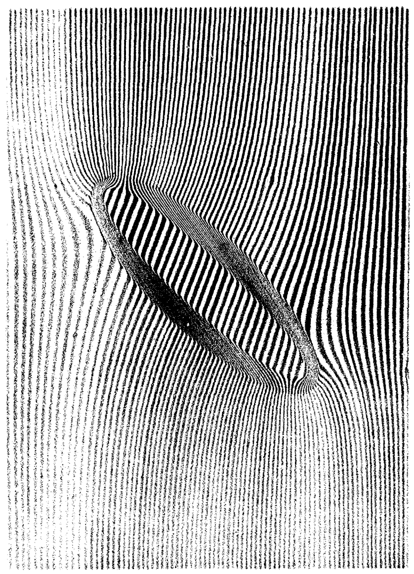


23

Flat Elliptic Plate, inclined 45° to field. Permeability = 100.



24

Hollow Elliptic Cylinder. Major axis inclined 45° to field. Permeability = 100.

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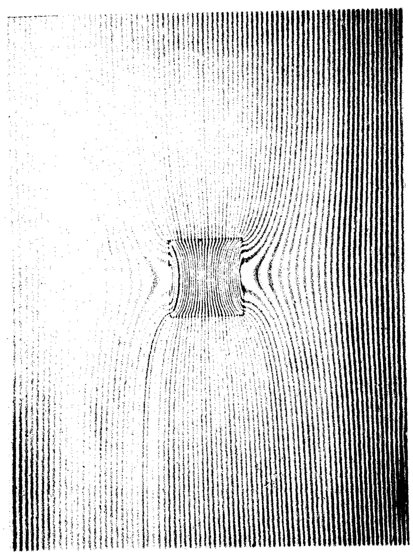
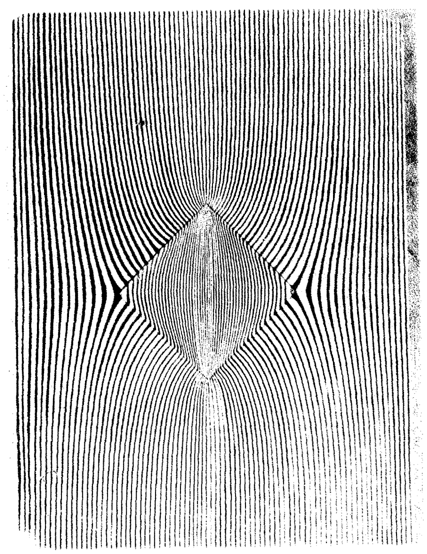


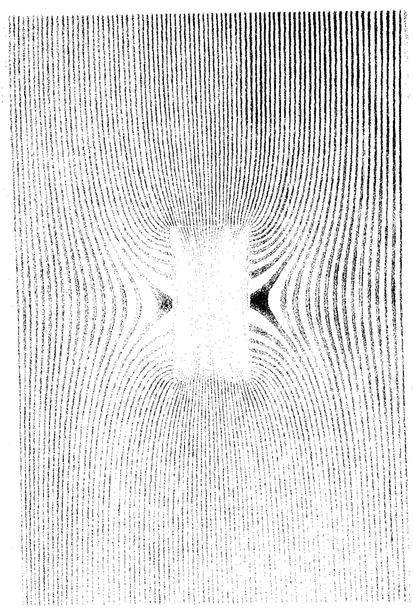
25

Infinite Cylinder of Square Section.
Permeability = 100.



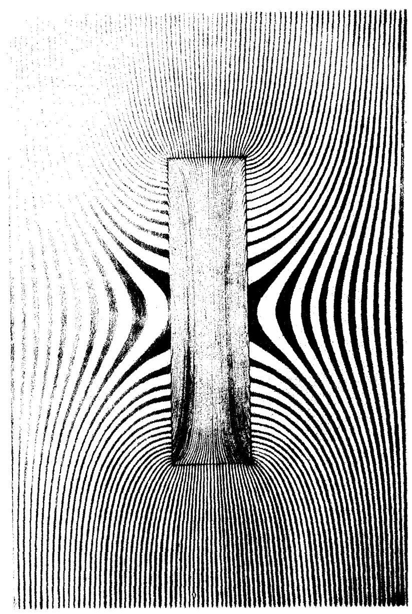
26

Infinite Cylinder of Square Section.
Permeability = 100.



27

Infinite Cylinder of Rectangular Section.
Permeability = 100.



28

Infinite Cylinder of Rectangular Section.
Permeability = 100.

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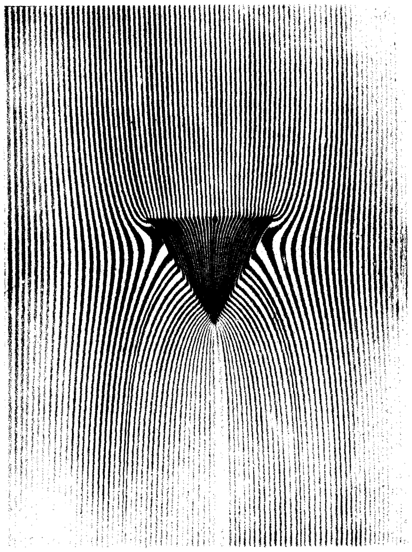
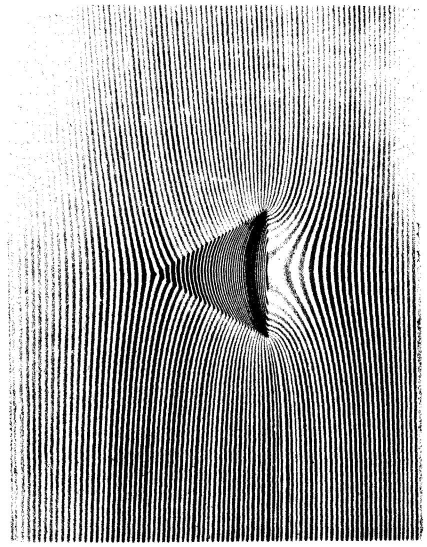


29

Infinite Cylinder of Triangular Section.
Permeability = 100.



30

Infinite Cylinder of Triangular Section.
Permeability = 100.

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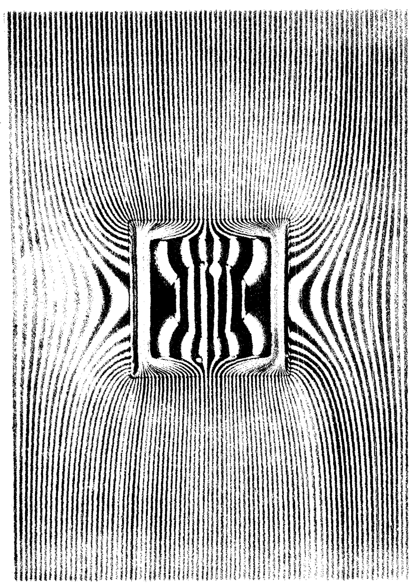
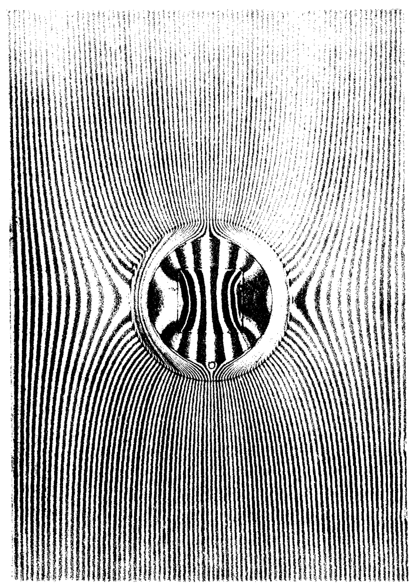


31

Hollow Square Cylinder enclosing Circular Solid one. Permeability = 100.



32

Hollow Circular Cylinder enclosing Solid Square one. Permeability = 100.

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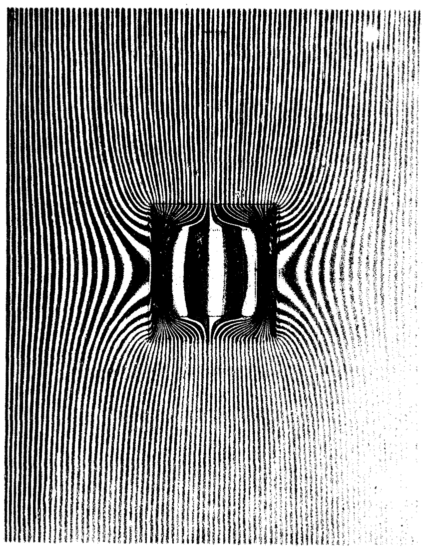
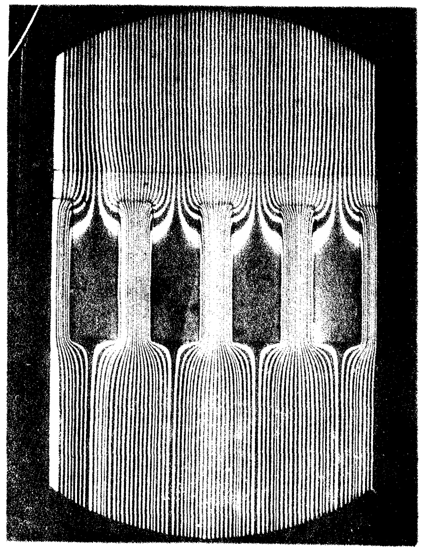


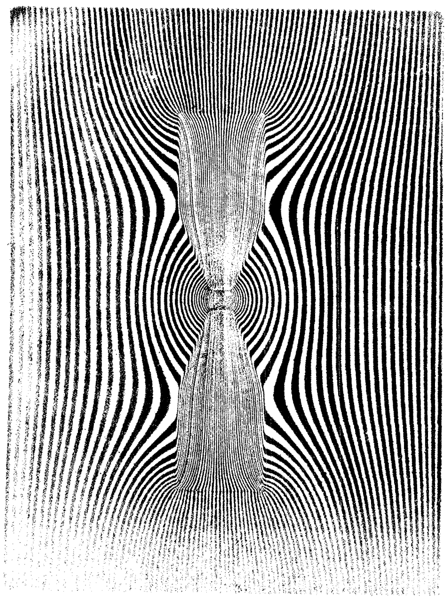
33

Hollow Square Shield. Permeability=100.



34

Induction in Air-gap and Teeth of Toothed-core Armature. Permeability=100.



35

Field between Tapered Pole-pieces of Electromagnet.

36

Attraction of Armature by Pole-piece.

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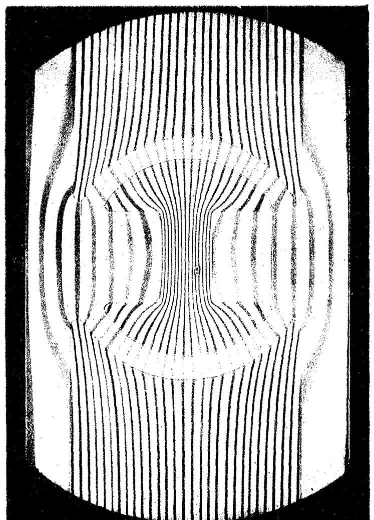
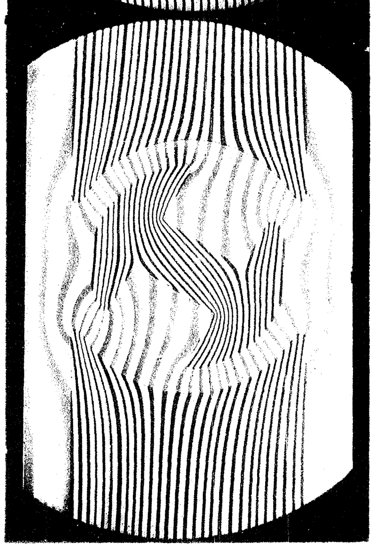
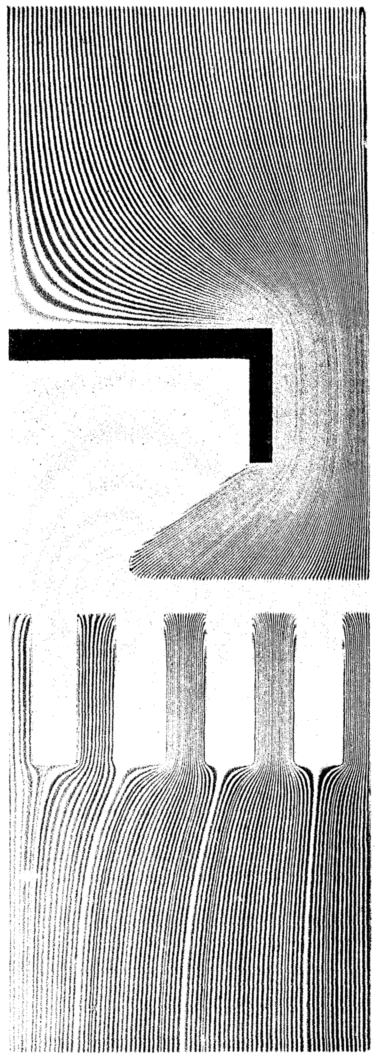
37





38

Siemens Shuttle-wound Armature in two positions.



39

Fringe of Magnetic Field near edge of Pole-piece in Dynamo with Toothed-core Armature.

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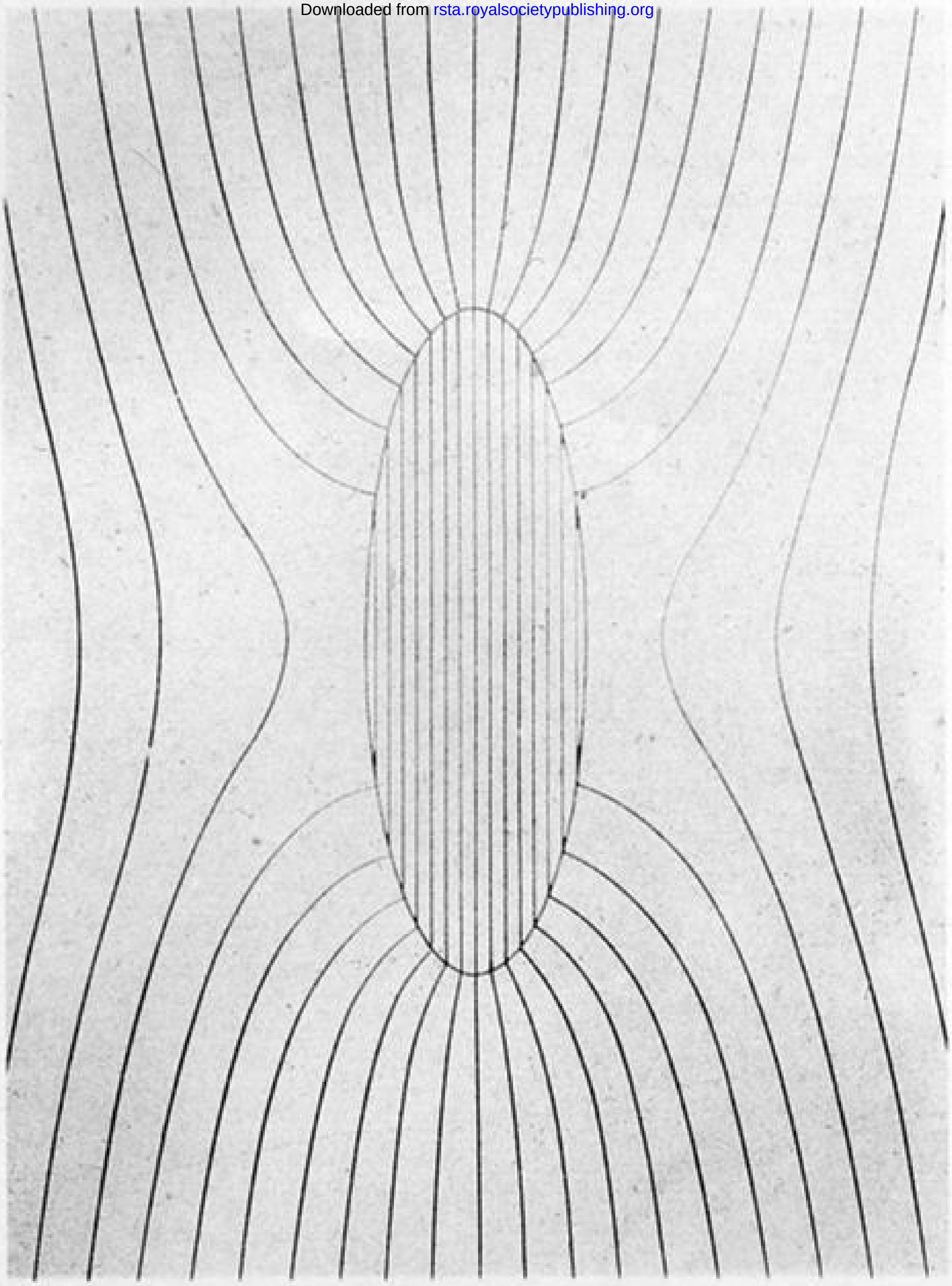

9

heoretical Diagram for Infinite Elliptic Cylinder placed in uniform field. Ratio of axes 3 : 1. Permeability = 100.

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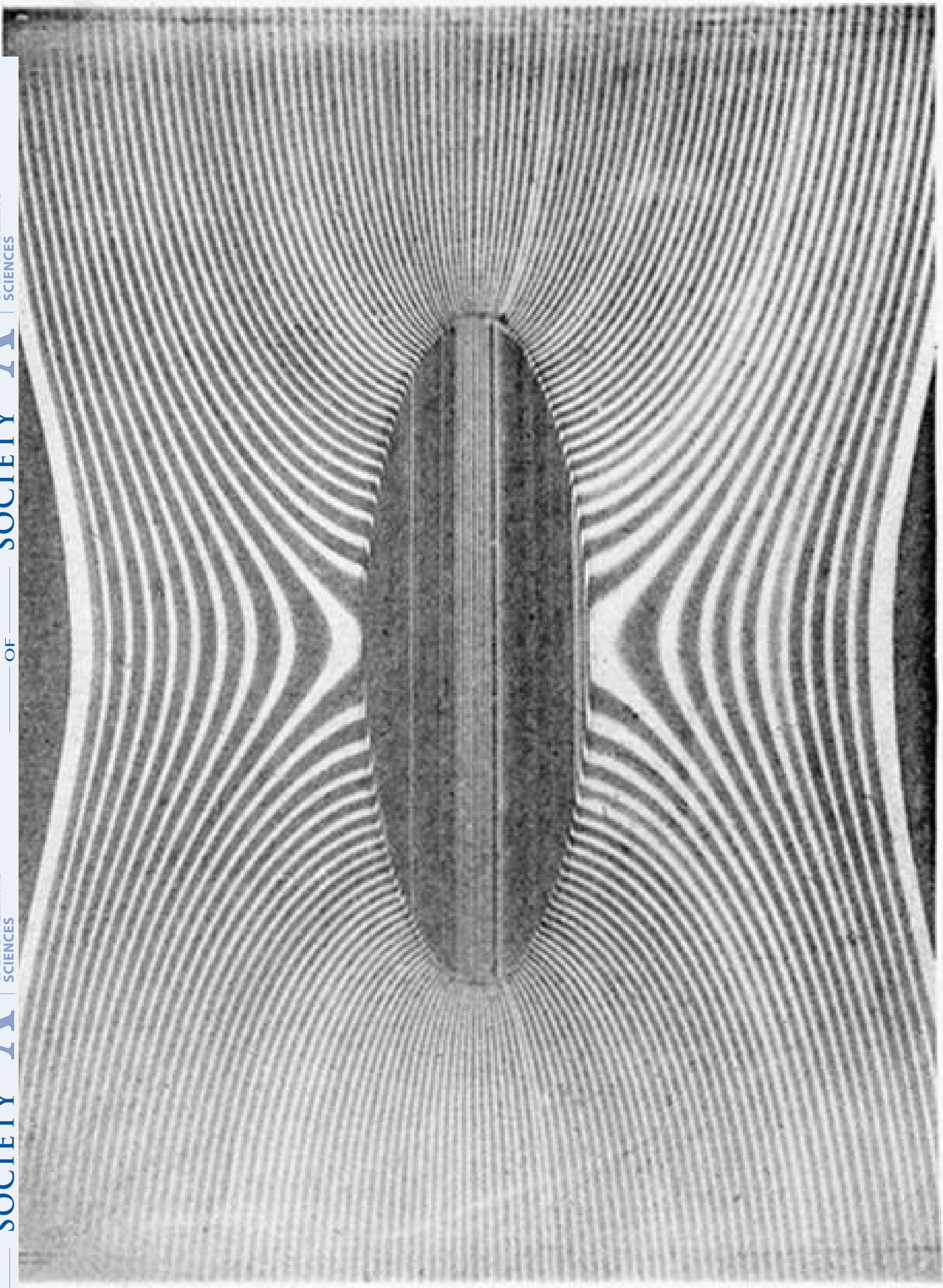

10

tream-line Diagram corresponding to theoretical Diagram of Fig. 9.

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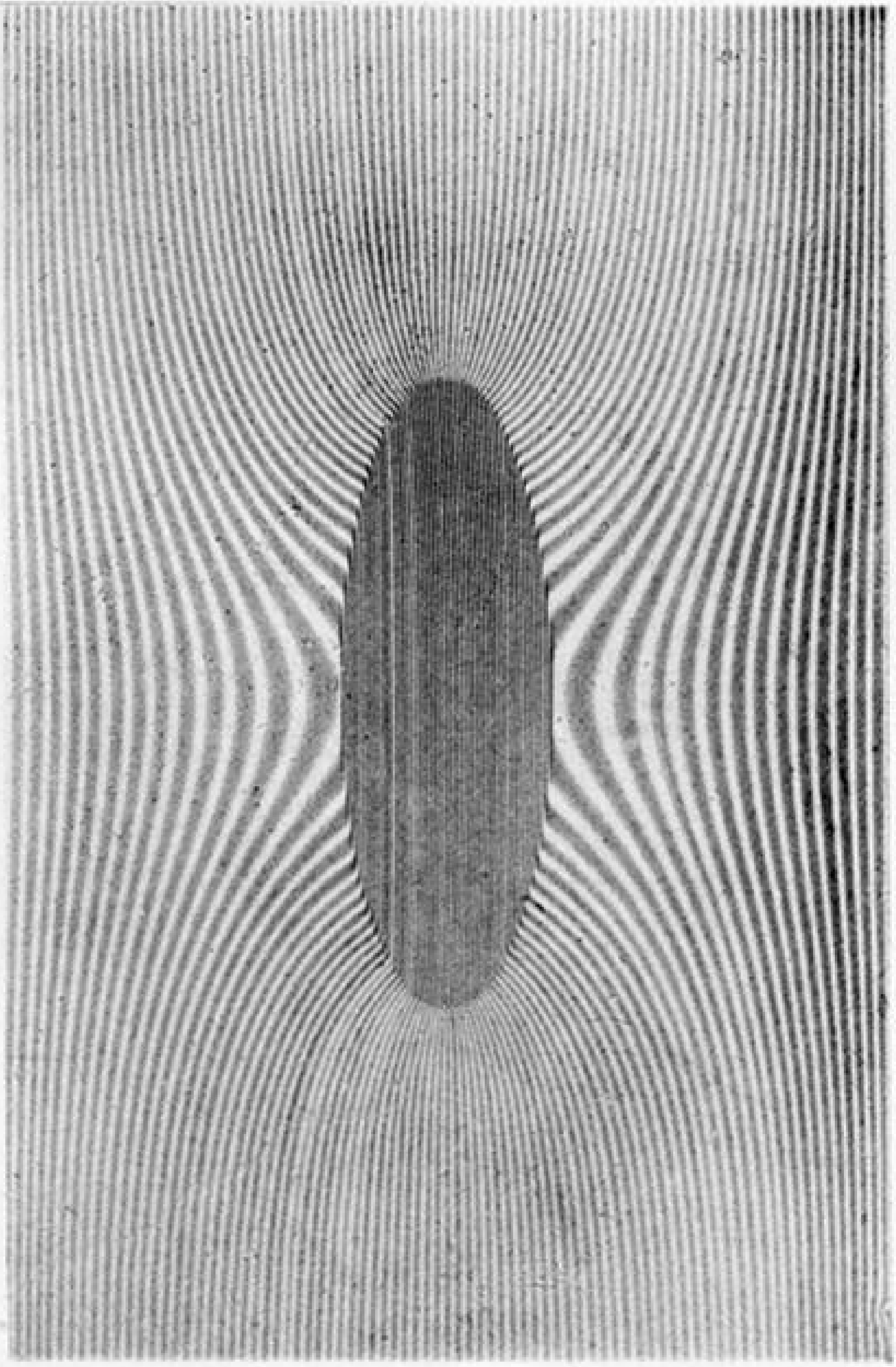

11

ıfinite Elliptic Cylinder in uniform field.
Ratio of axes 3 : 1. Permeability = 20.

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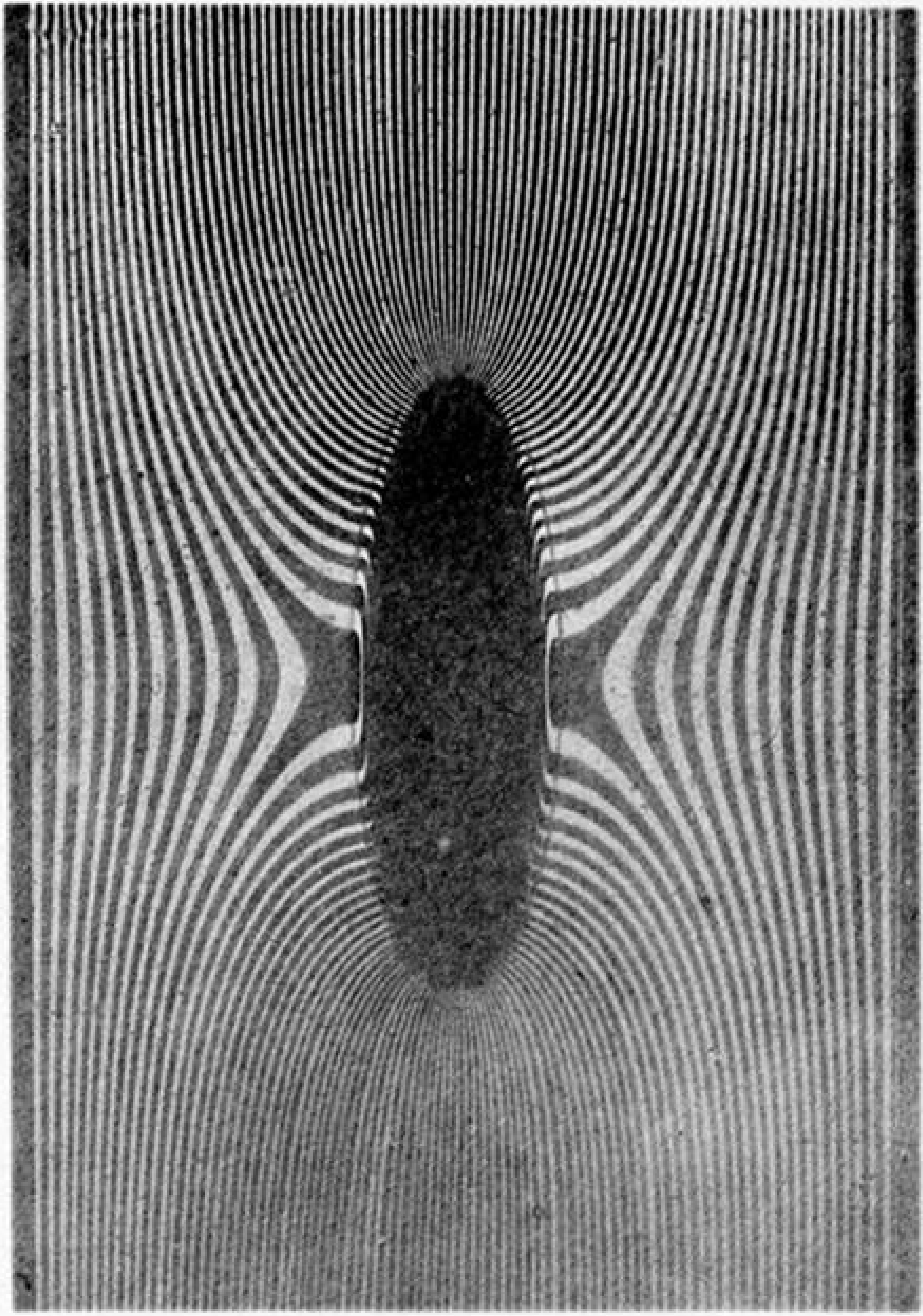

12

Infinite Elliptic Cylinder in uniform field.
Ratio of axes 3 : 1. Permeability 1000.

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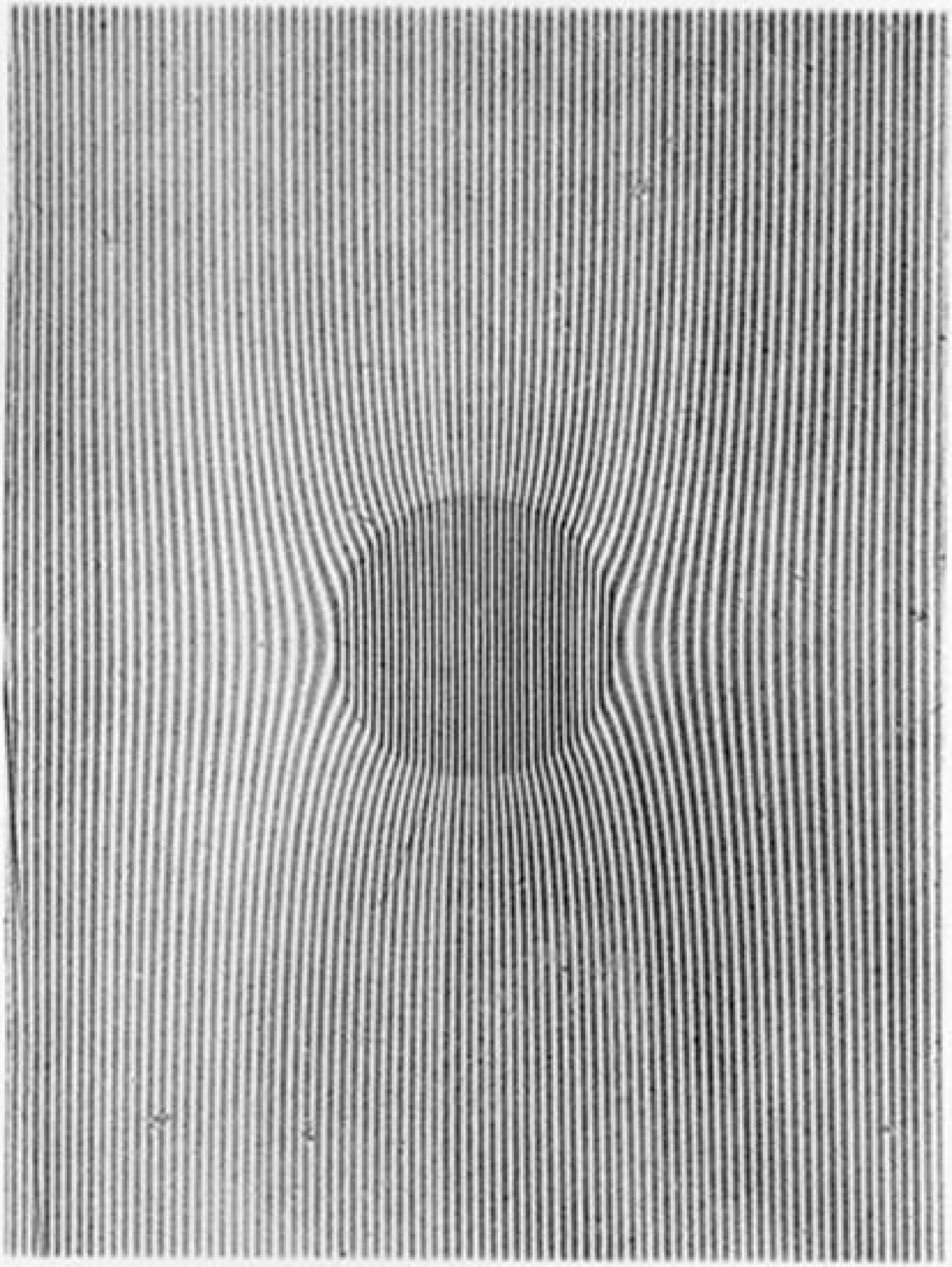
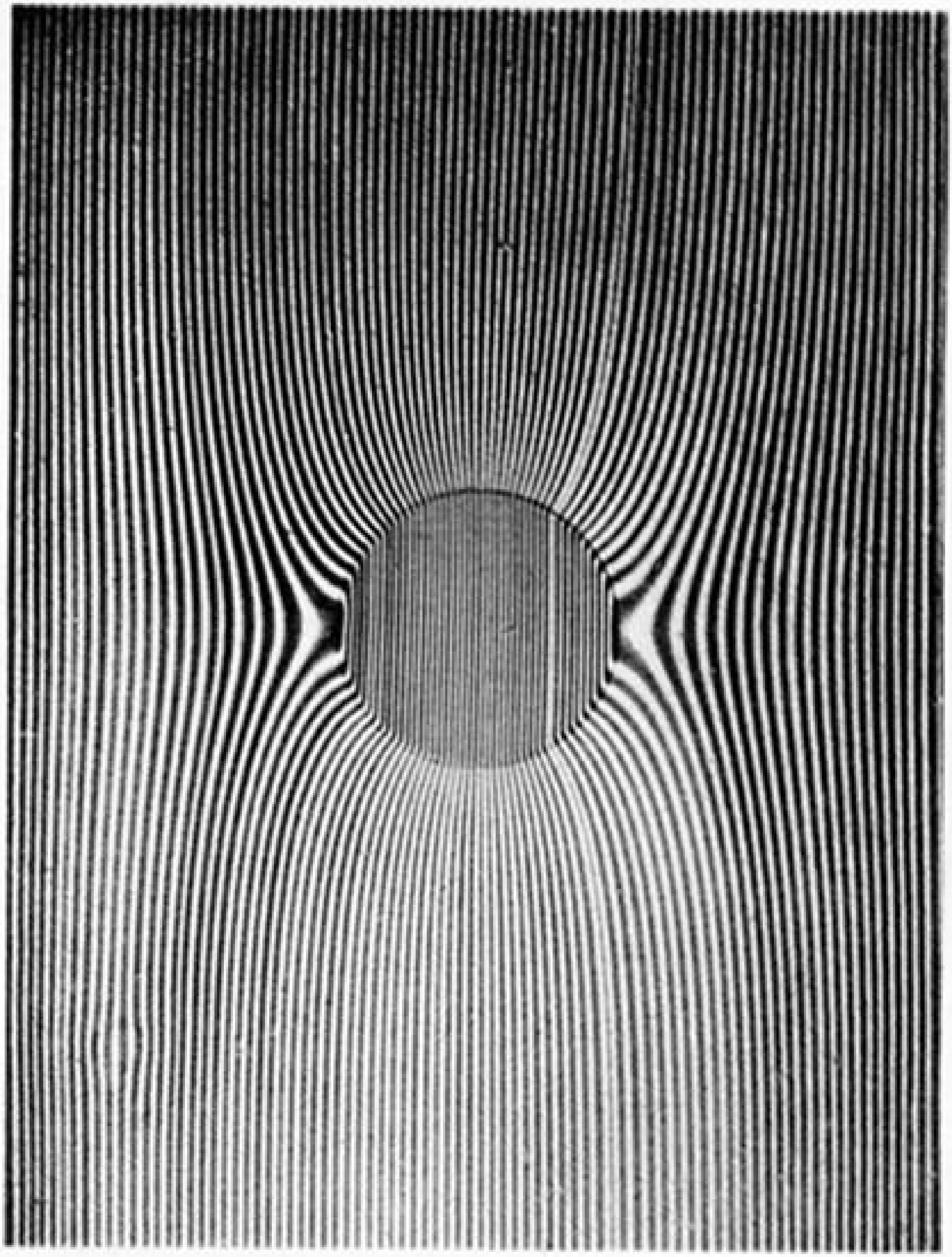


13

…nfinite Circular Cylinder in uniform field.
Permeability=2.



14

nfinite Circular Cylinder in uniform field.
Permeability=100.

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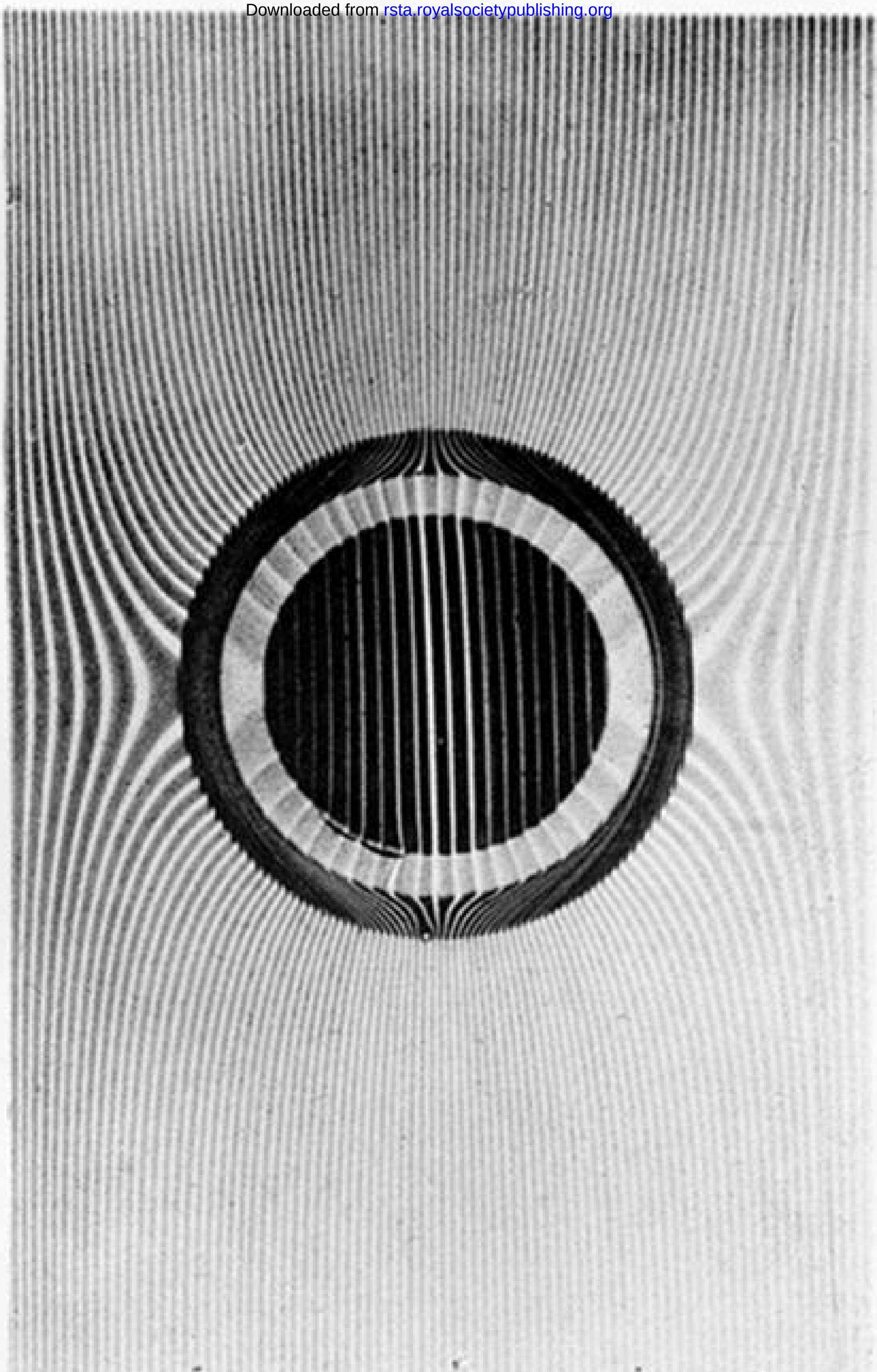

15

olid Circular Cylinder shielded by hollow one. Permeability=100.

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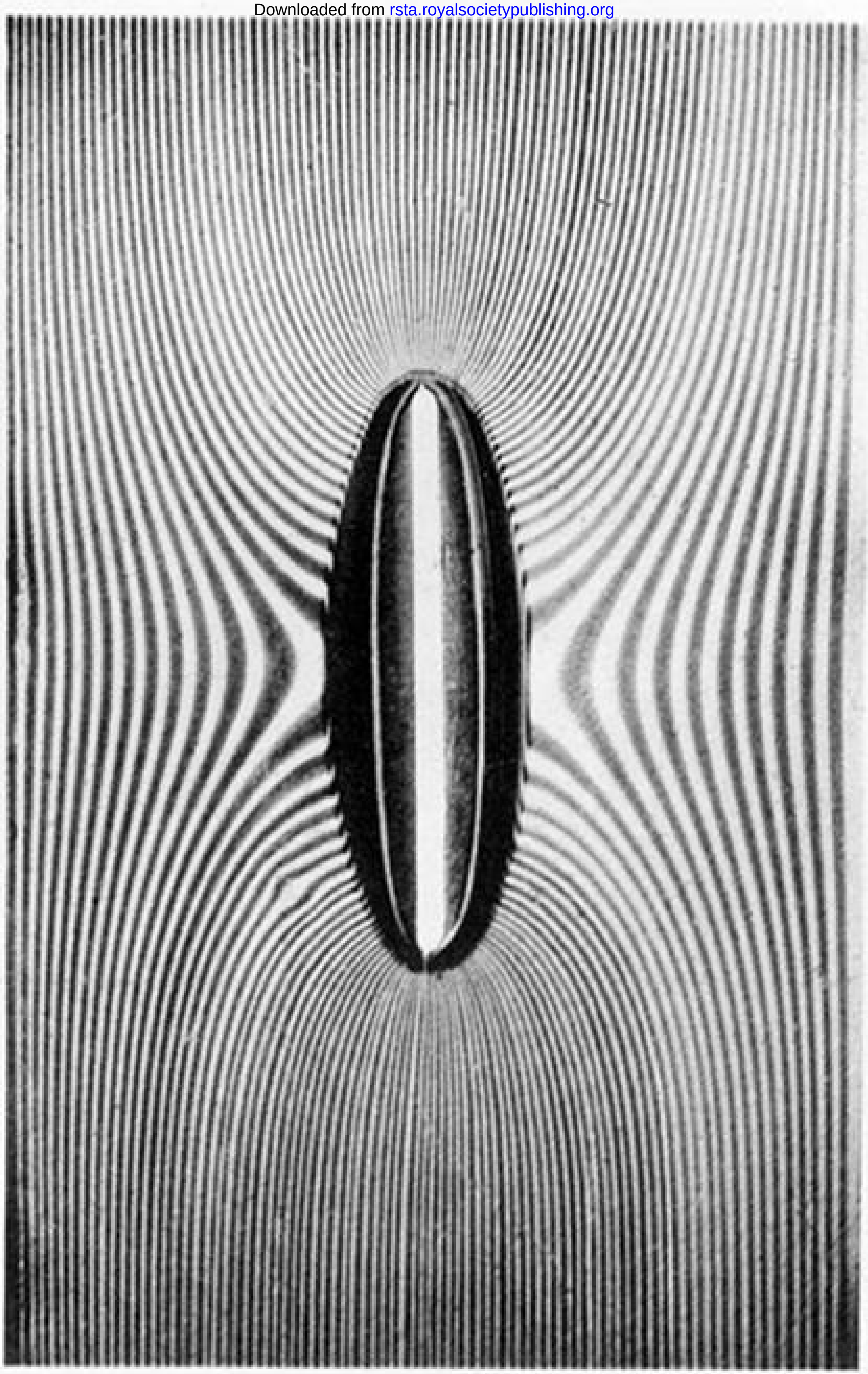

16

ıfinite Hollow Elliptic Cylinder in uniform field. Permeability = 100.

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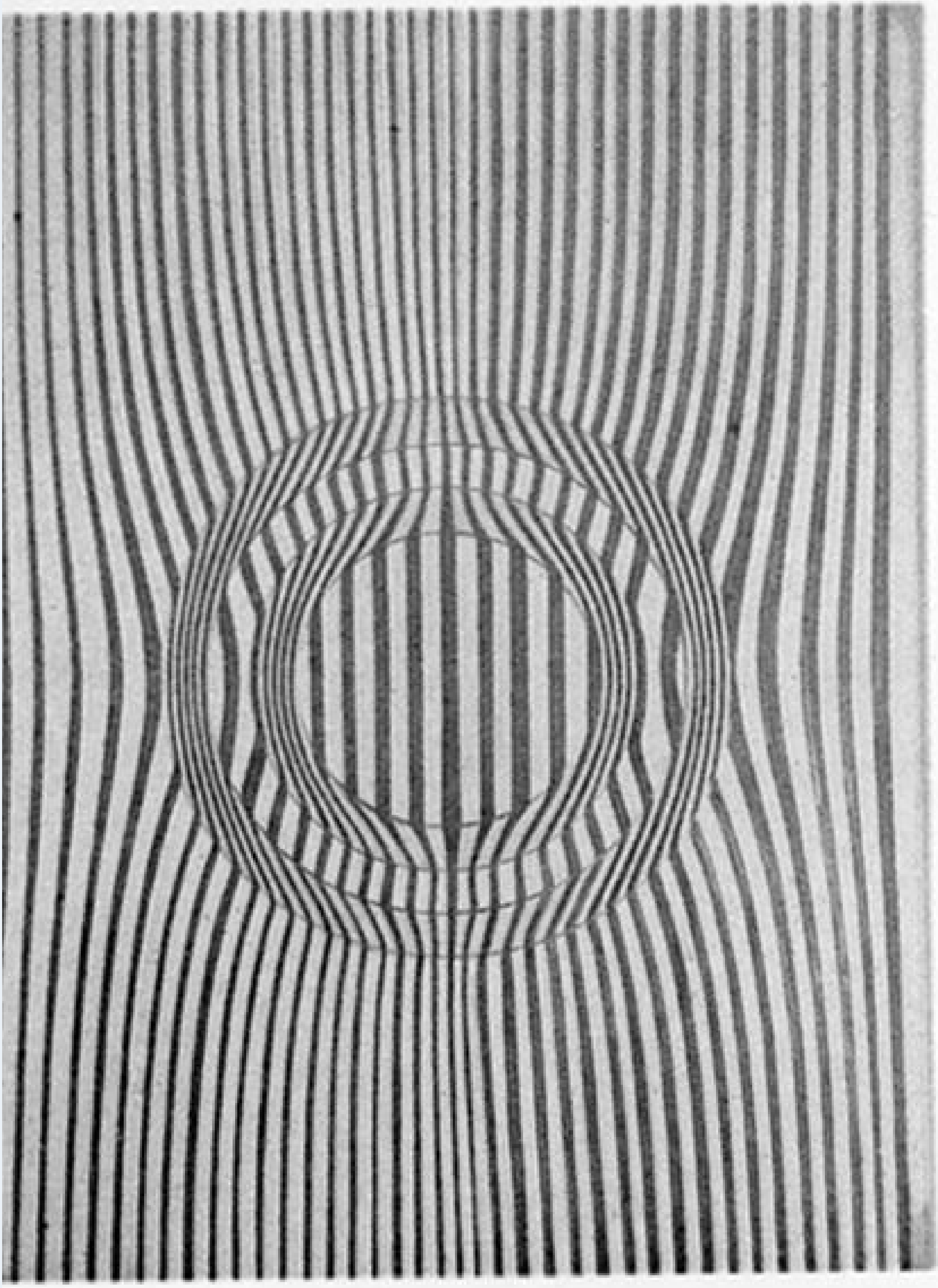

17

Double Cylindric Shield of very low permeability.

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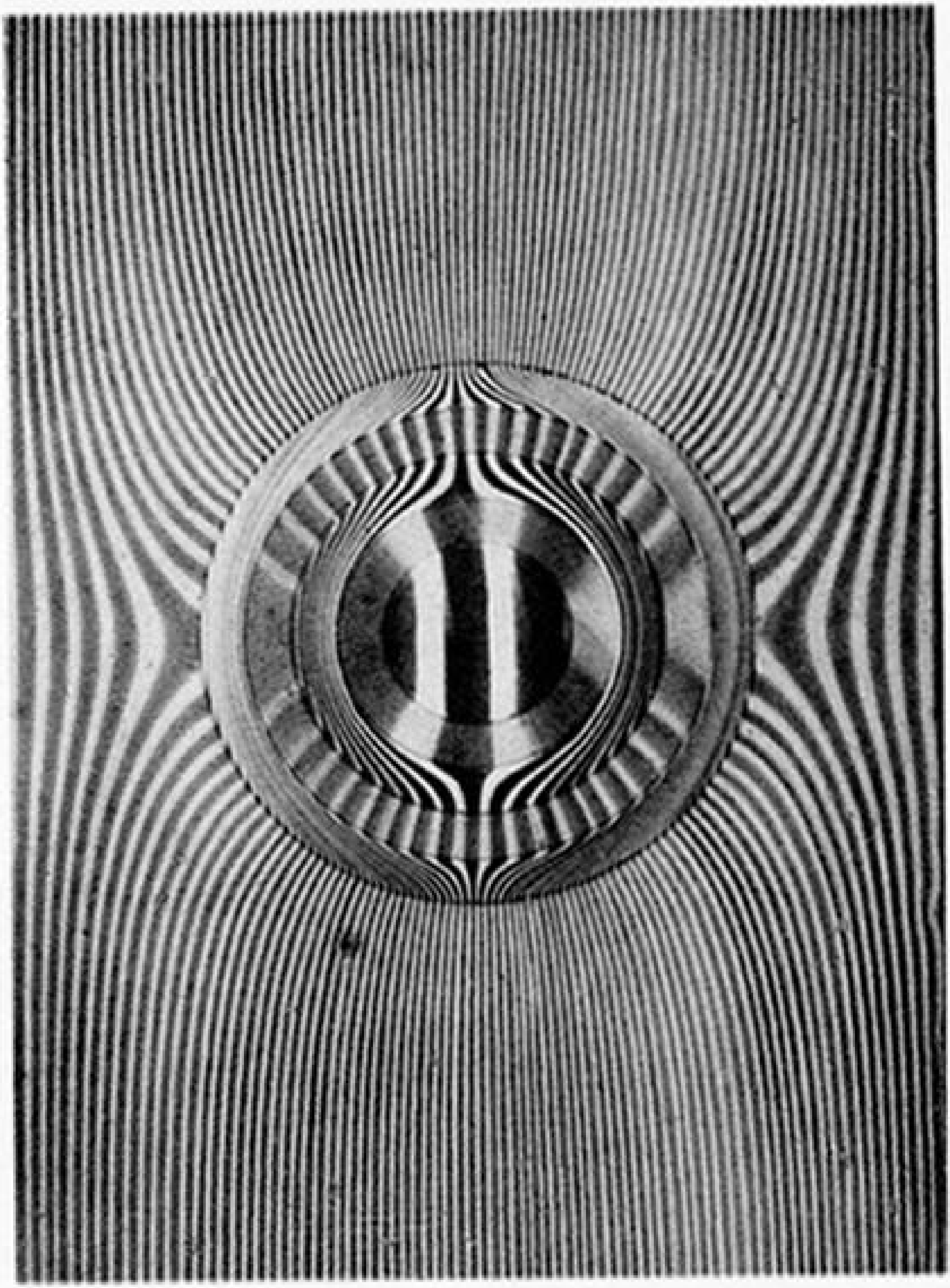
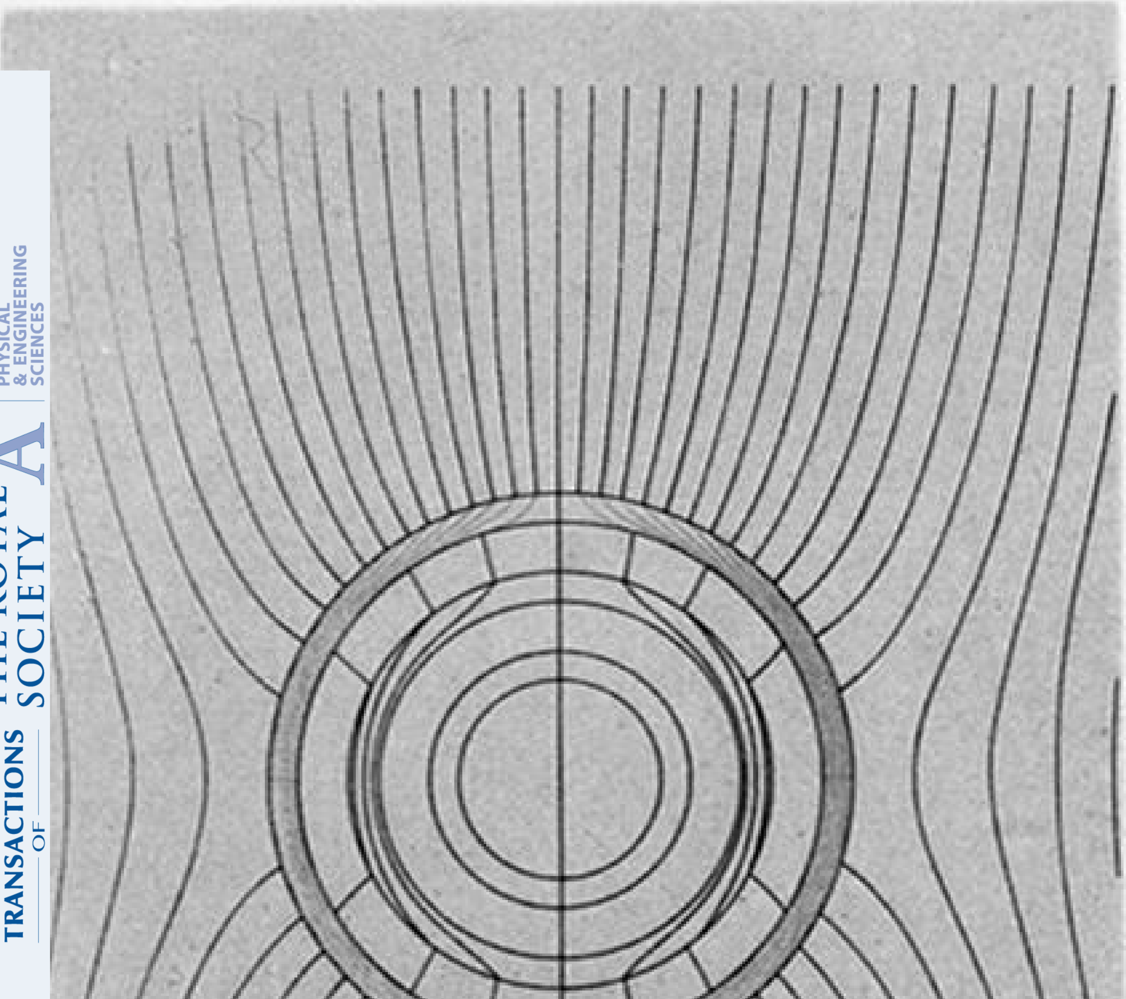


18

ouble Cylindric Shield enclosing Solid Cylinder. Permeability=100.



19

Triple Circular Cylindric Shield.
Theoretical Diagram.
Permeability = 100.

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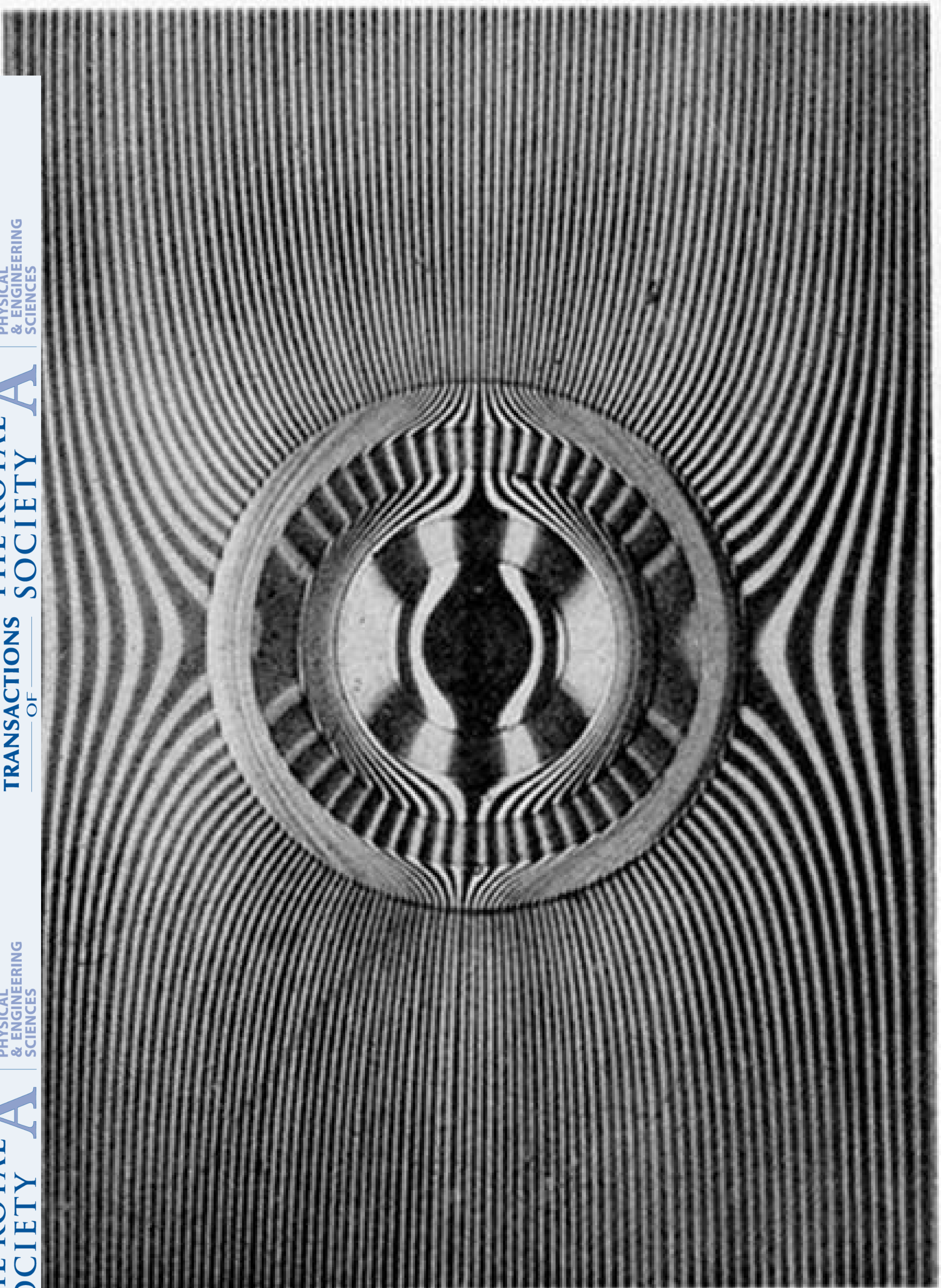

20

Triple Circular Cylindric Shield.
Stream-line Diagram.
Permeability=100.

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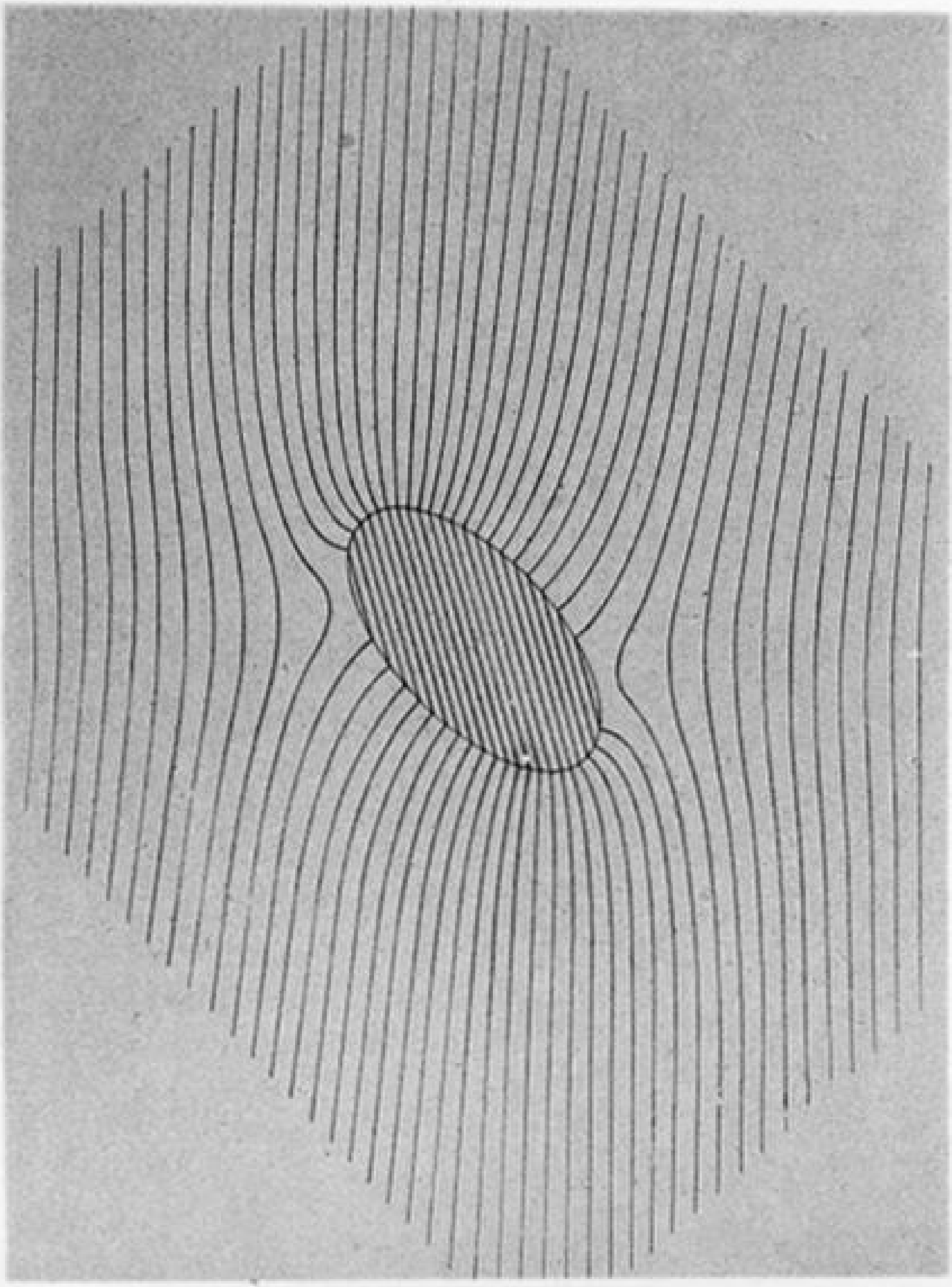

21

lliptic Cylinder. Ratio of axes 2 : 1, major axis inclined at 45° to field. Theoretical Diagram. Permeability=100.

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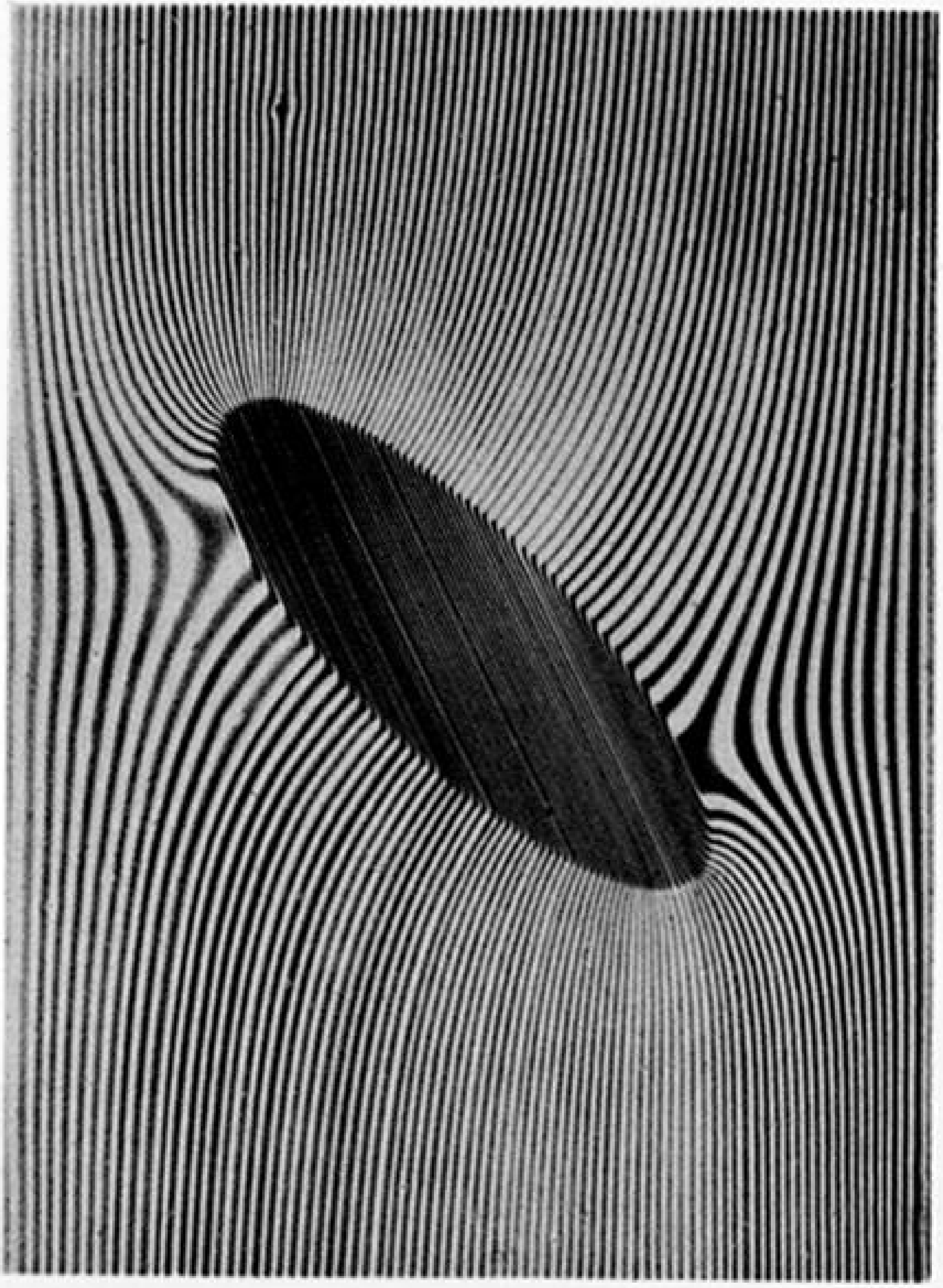

22

lliptic Cylinder. Ratio of axes 3 : 1, major axis inclined at 45° to field. Stream-line Diagram. Permeability = 100.

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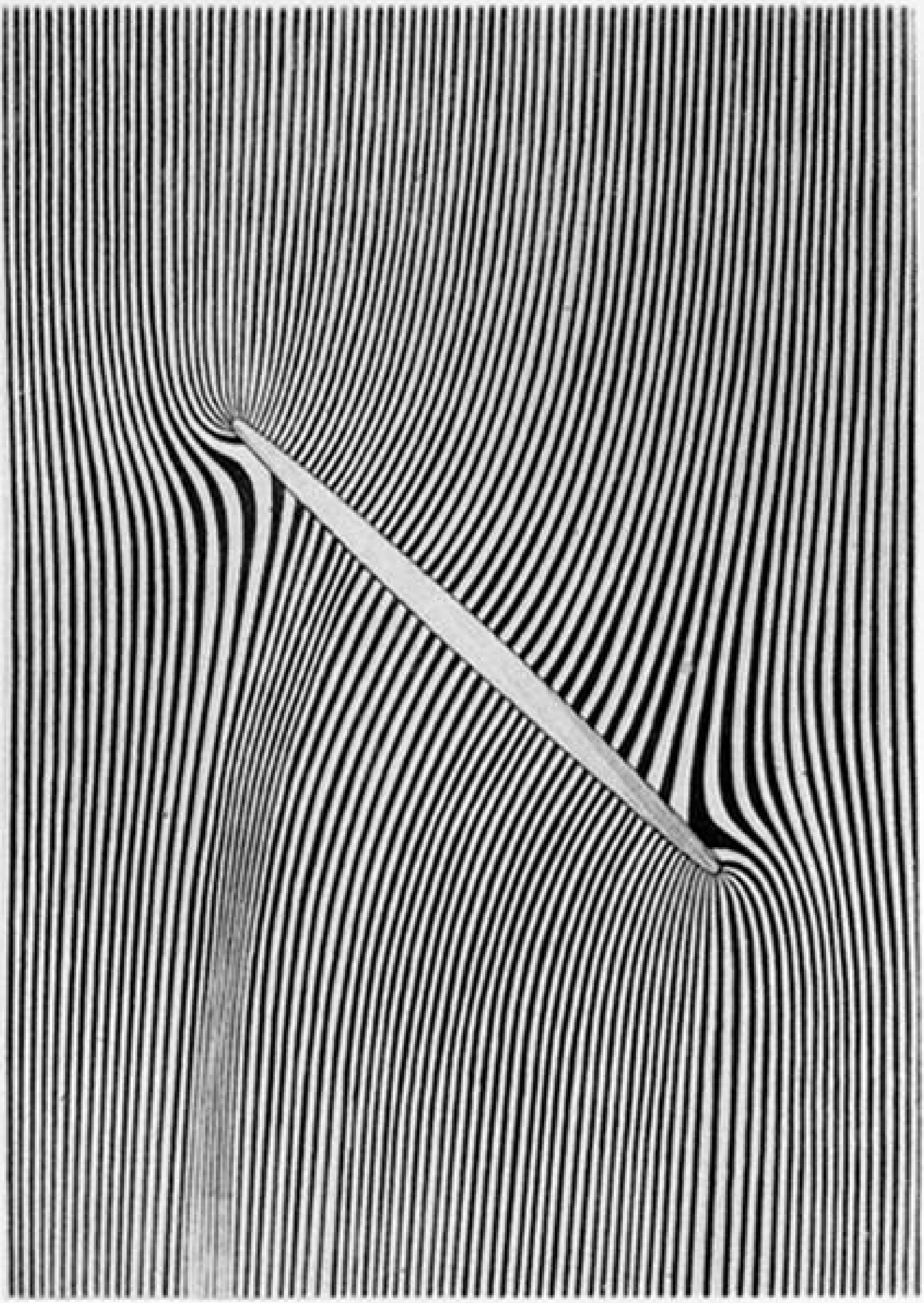

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23

lat Elliptic Plate, inclined 45° to field.
Permeability=100.

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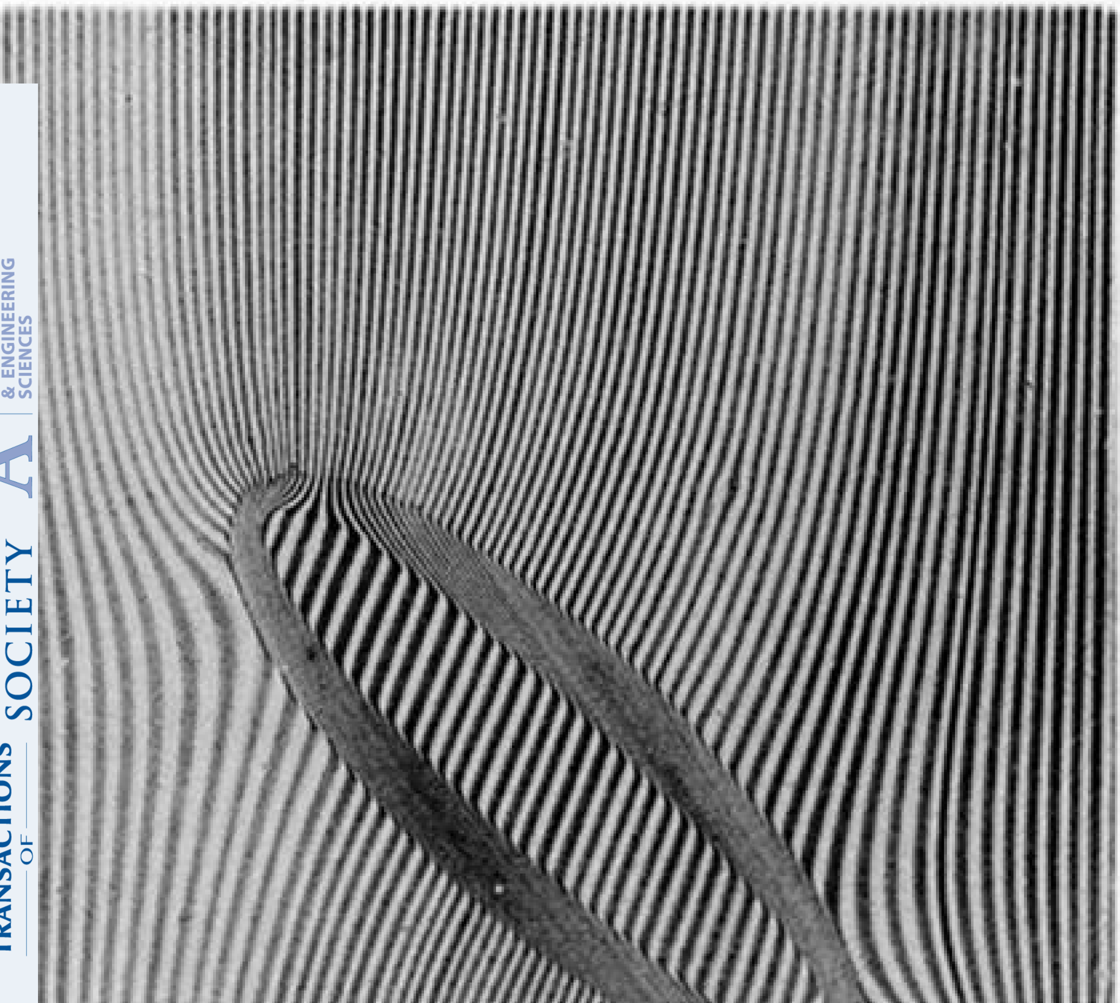

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24

Hollow Elliptic Cylinder.
Major axis inclined 45° to field.
Permeability = 100.

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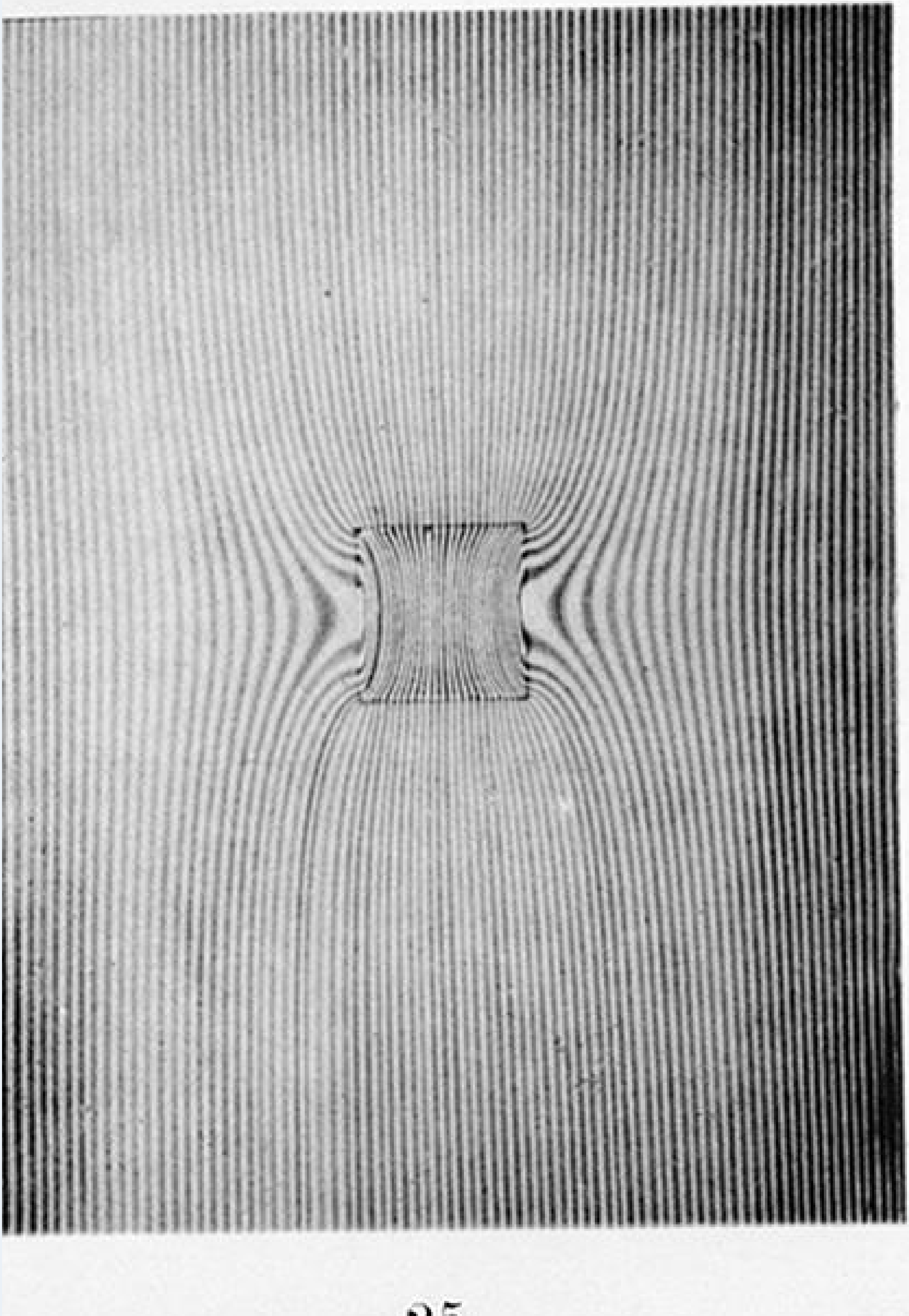

25

nfinite Cylinder of Square Section.
Permeability = 100.

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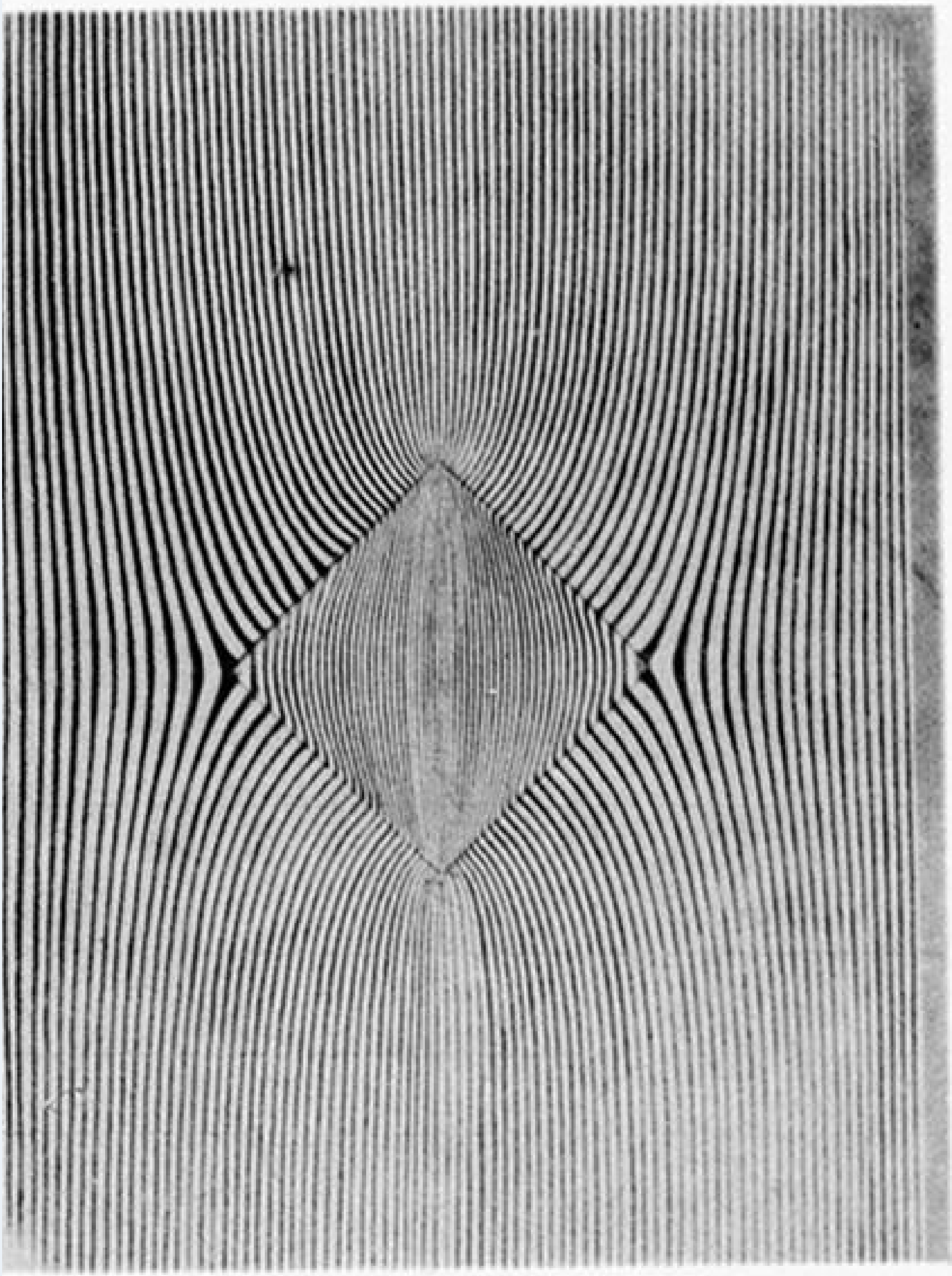

26

nfinite Cylinder of Square Section.
Permeability = 100.

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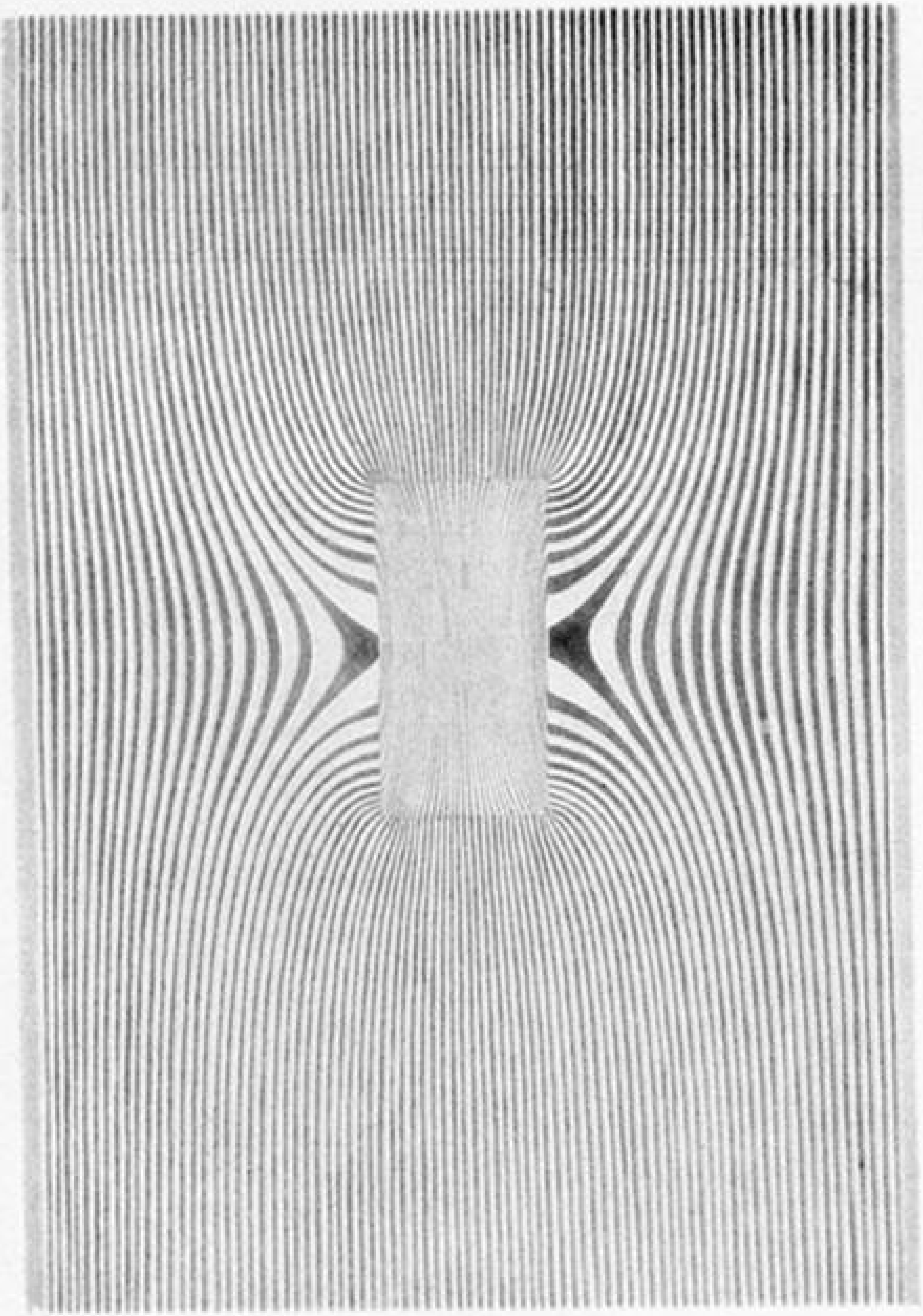

27

ıfinite Cylinder of Rectangular Section.
Permeability = 100.

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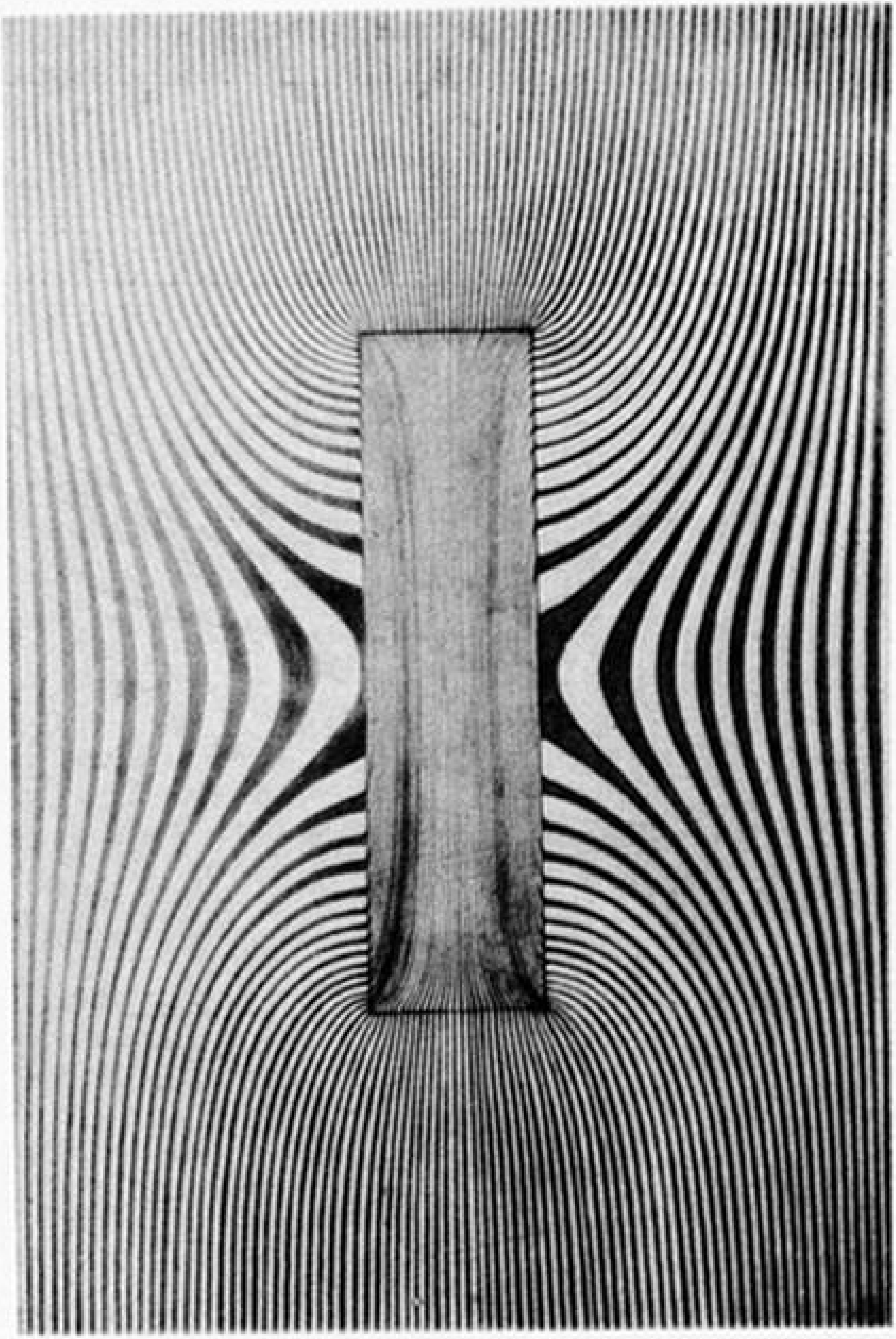

28

ıfinite Cylinder of Rectangular Section.
Permeability = 100.

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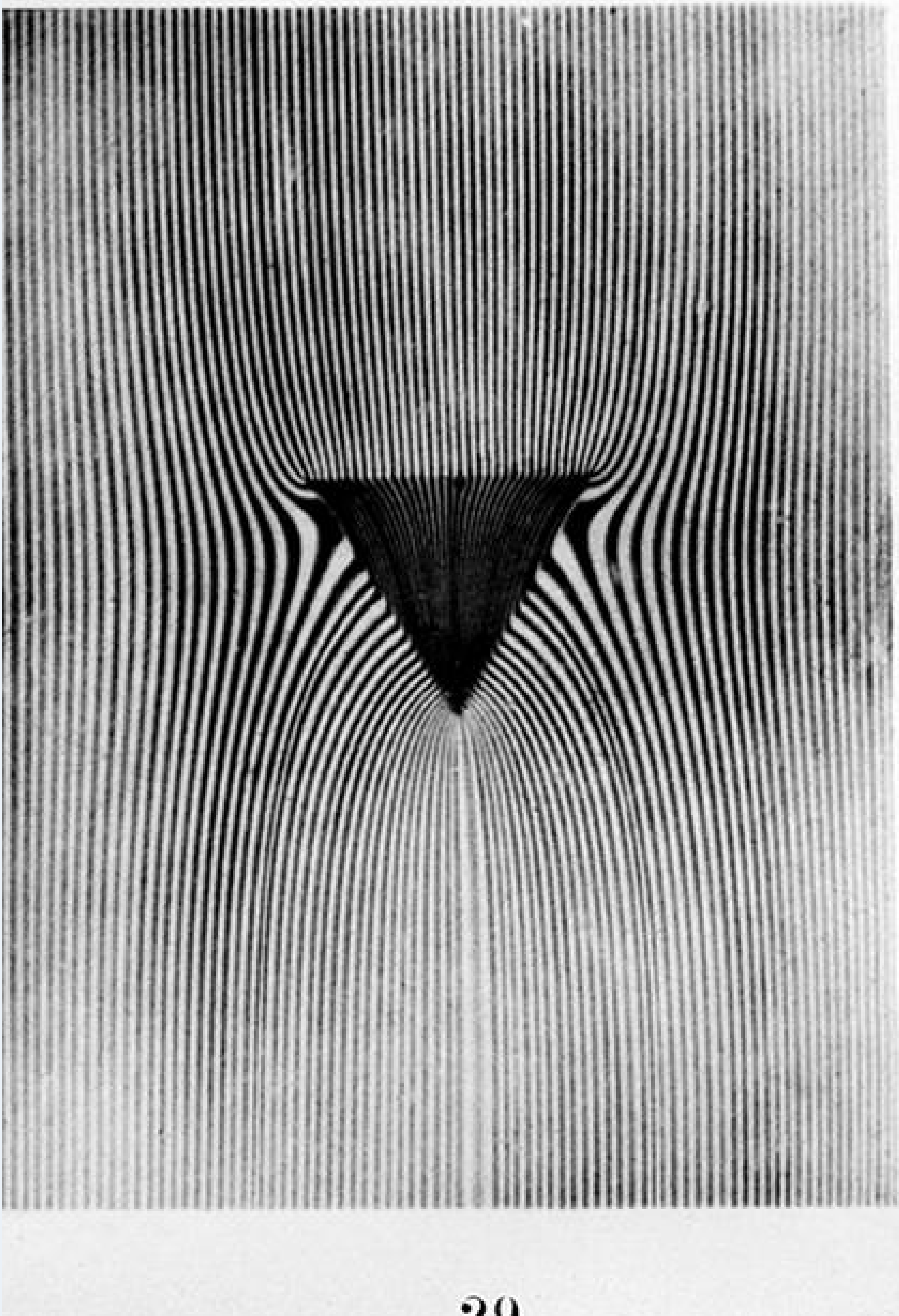
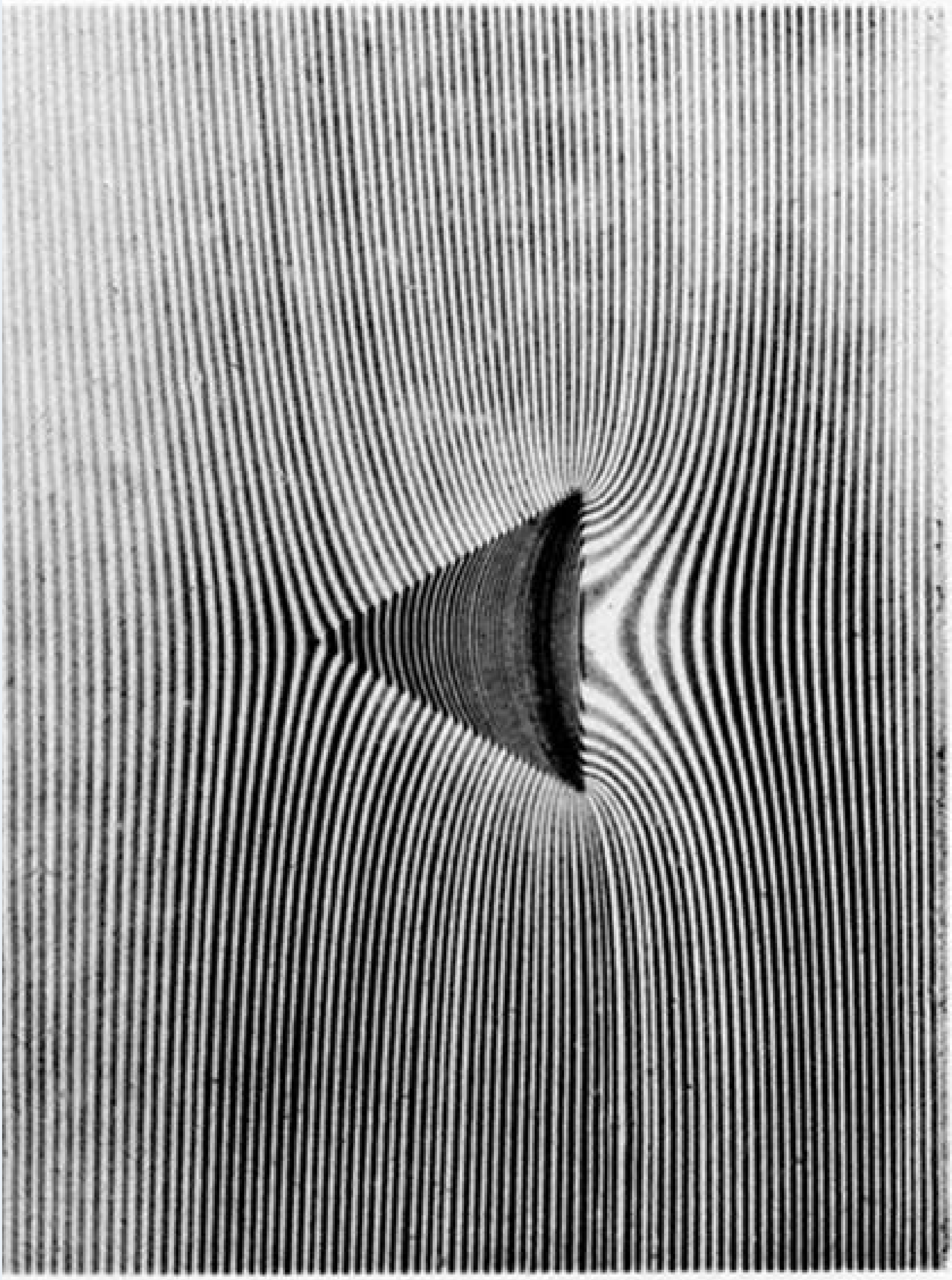


29

nfinite Cylinder of Triangular Section.
Permeability = 100.



30

nfinite Cylinder of Triangular Section.
Permeability=100.

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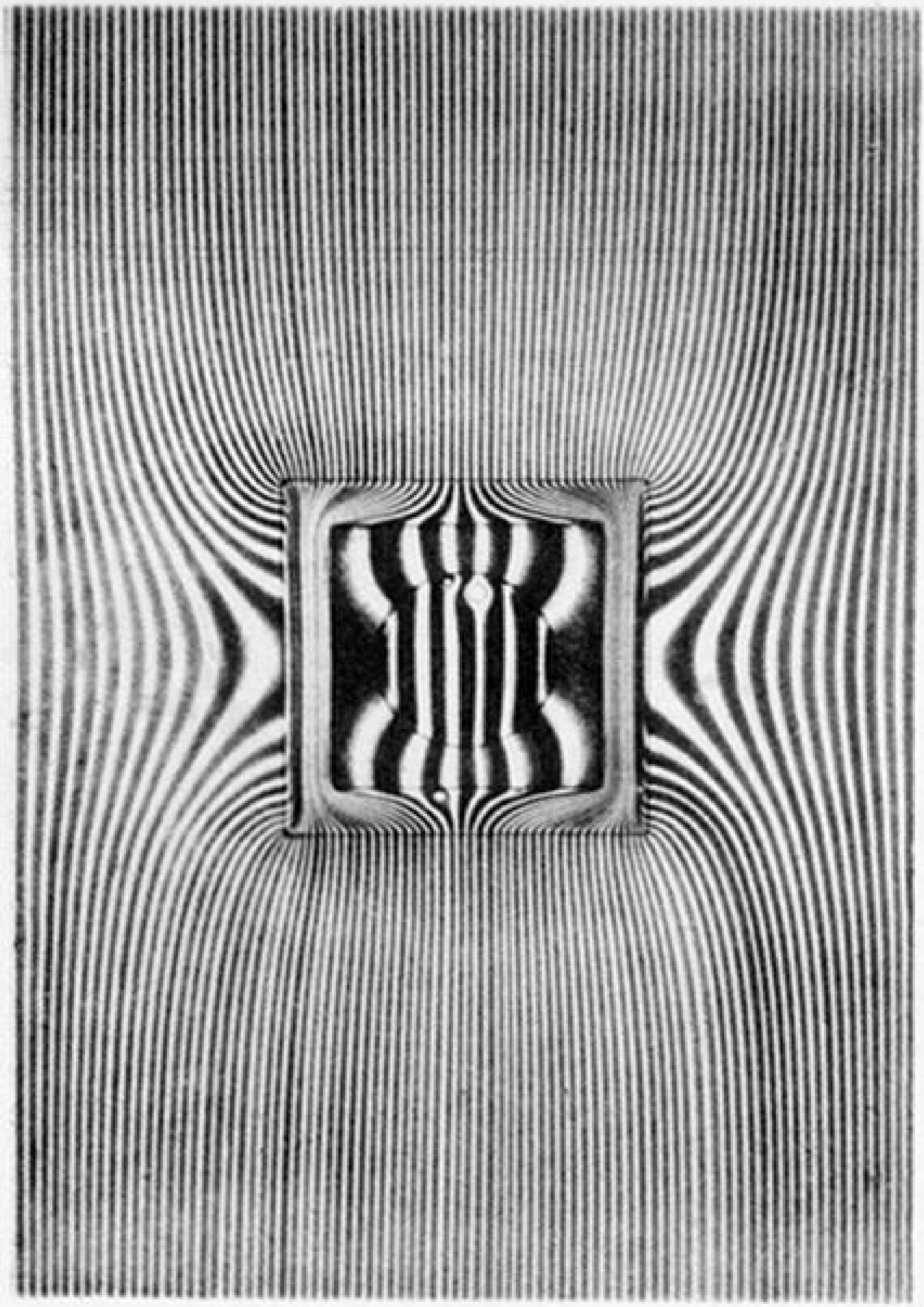

31

ollow Square Cylinder enclosing Circular
Solid one. Permeability=100.

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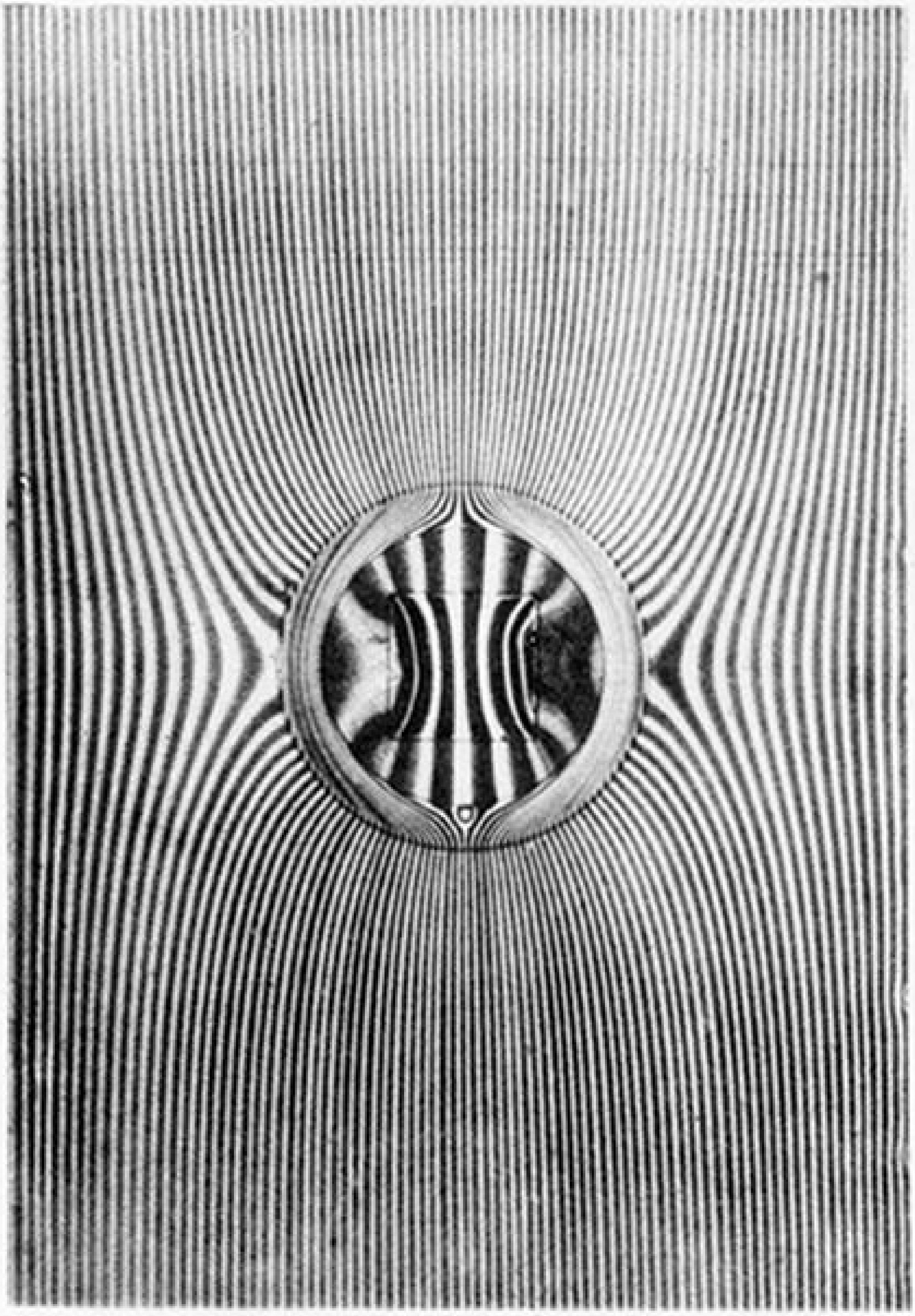

32

ollow Circular Cylinder enclosing Solid Square one. Permeability=100.

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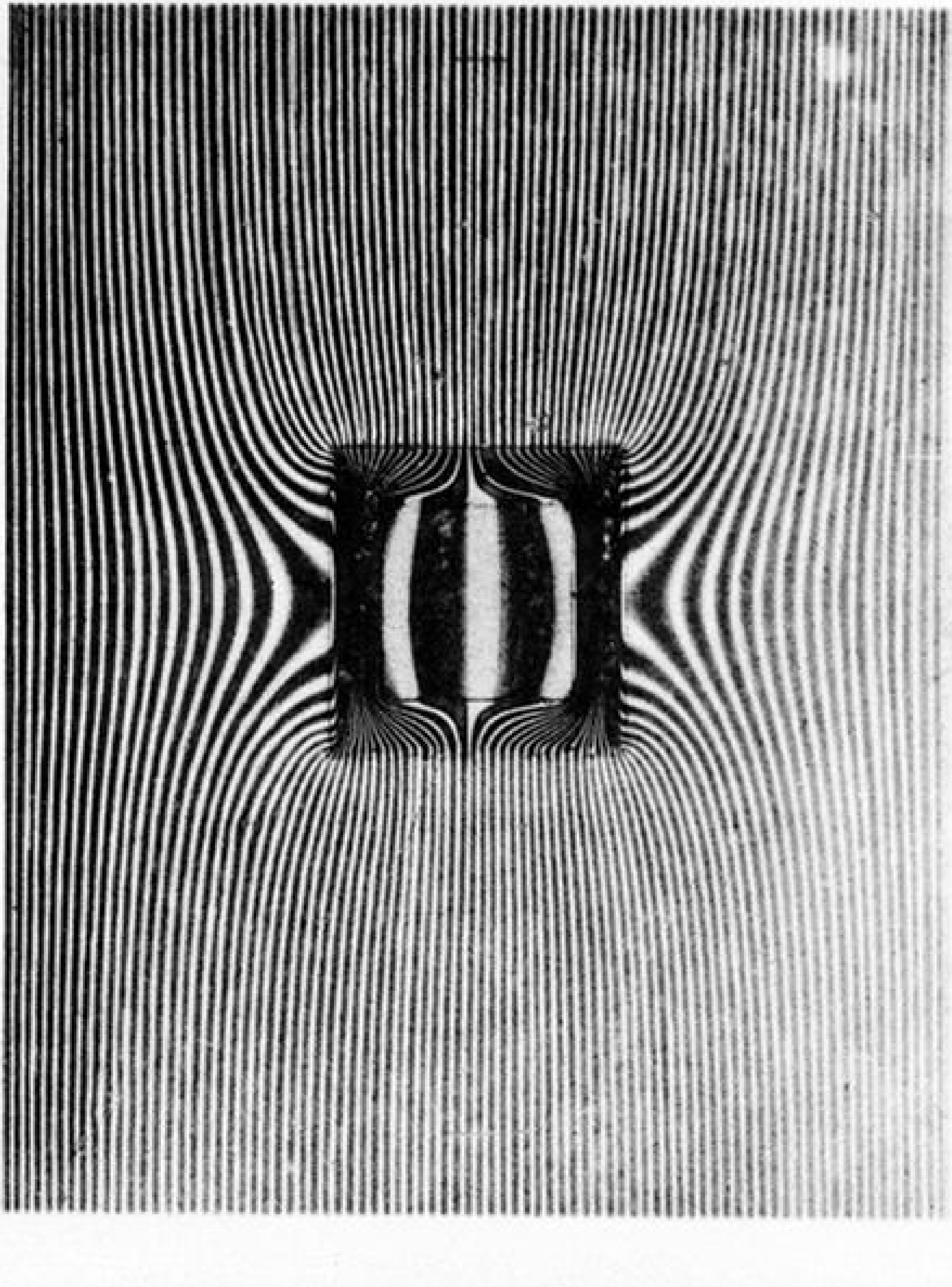

33

:ollow Square Shield. Permeability = 100.

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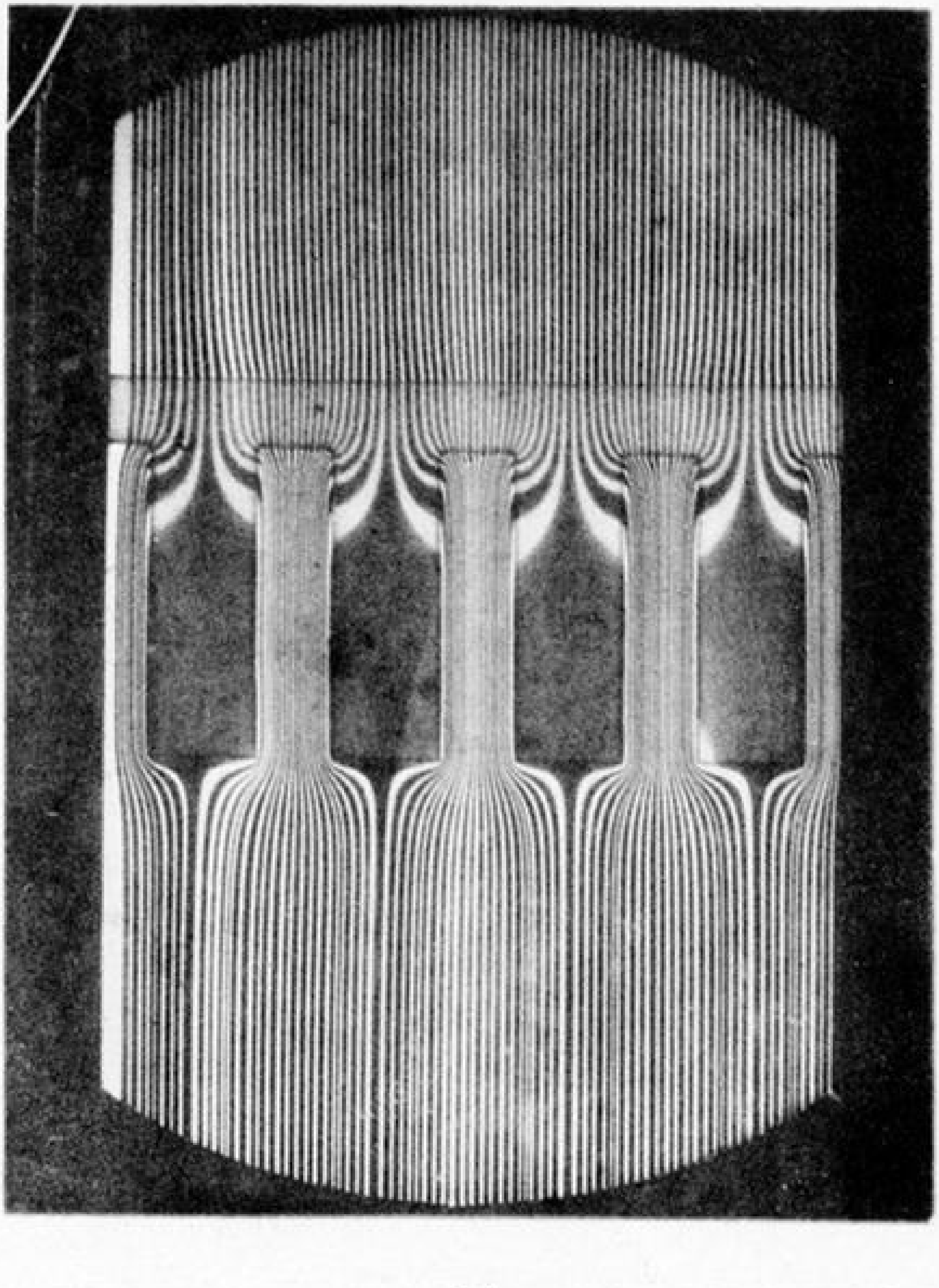

34

nduction in Air-gap and Teeth of Toothed-core Armature. Permeability=100.

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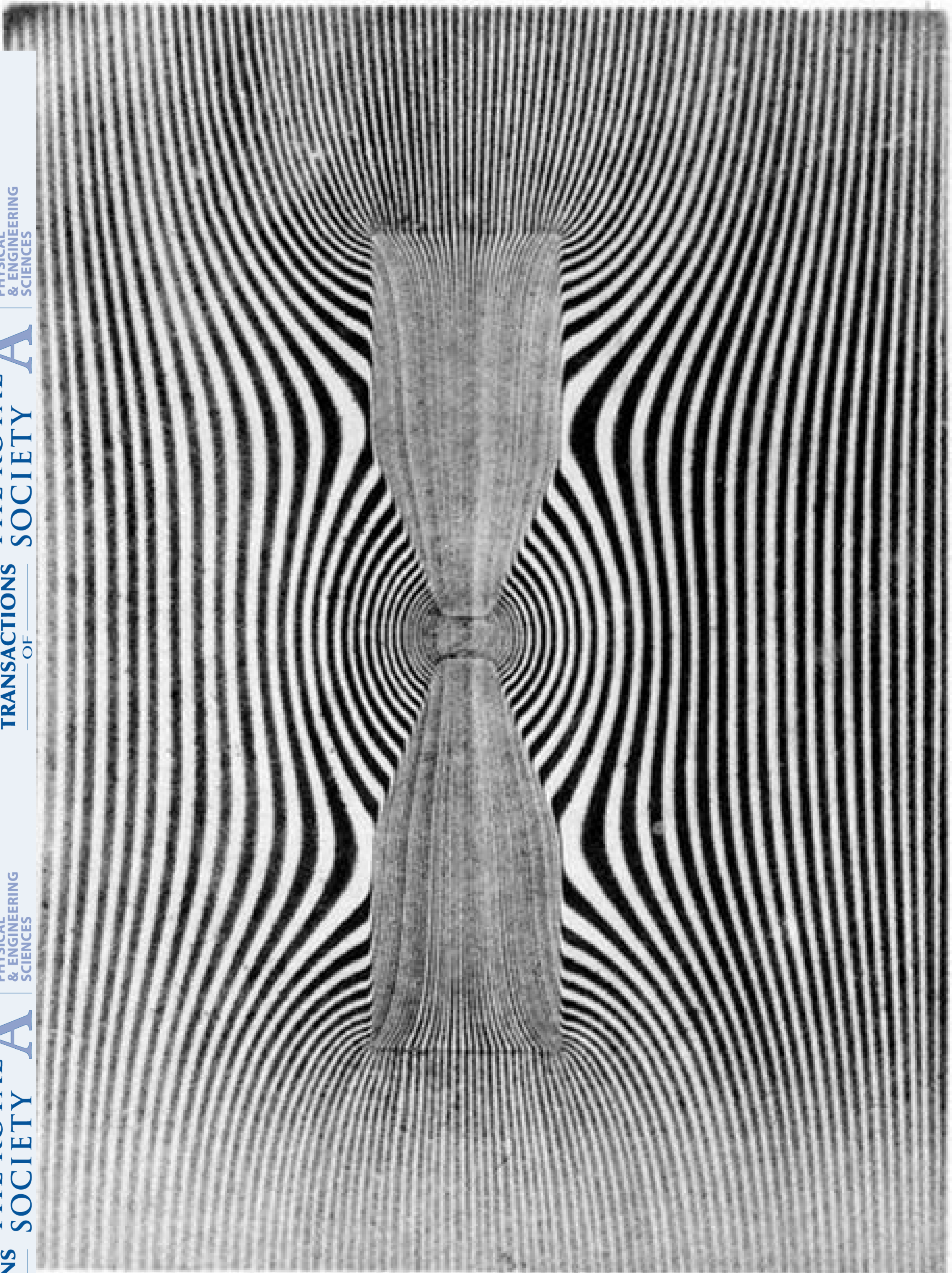

35

Field between Tapered Pole-pieces of Electromagnet.

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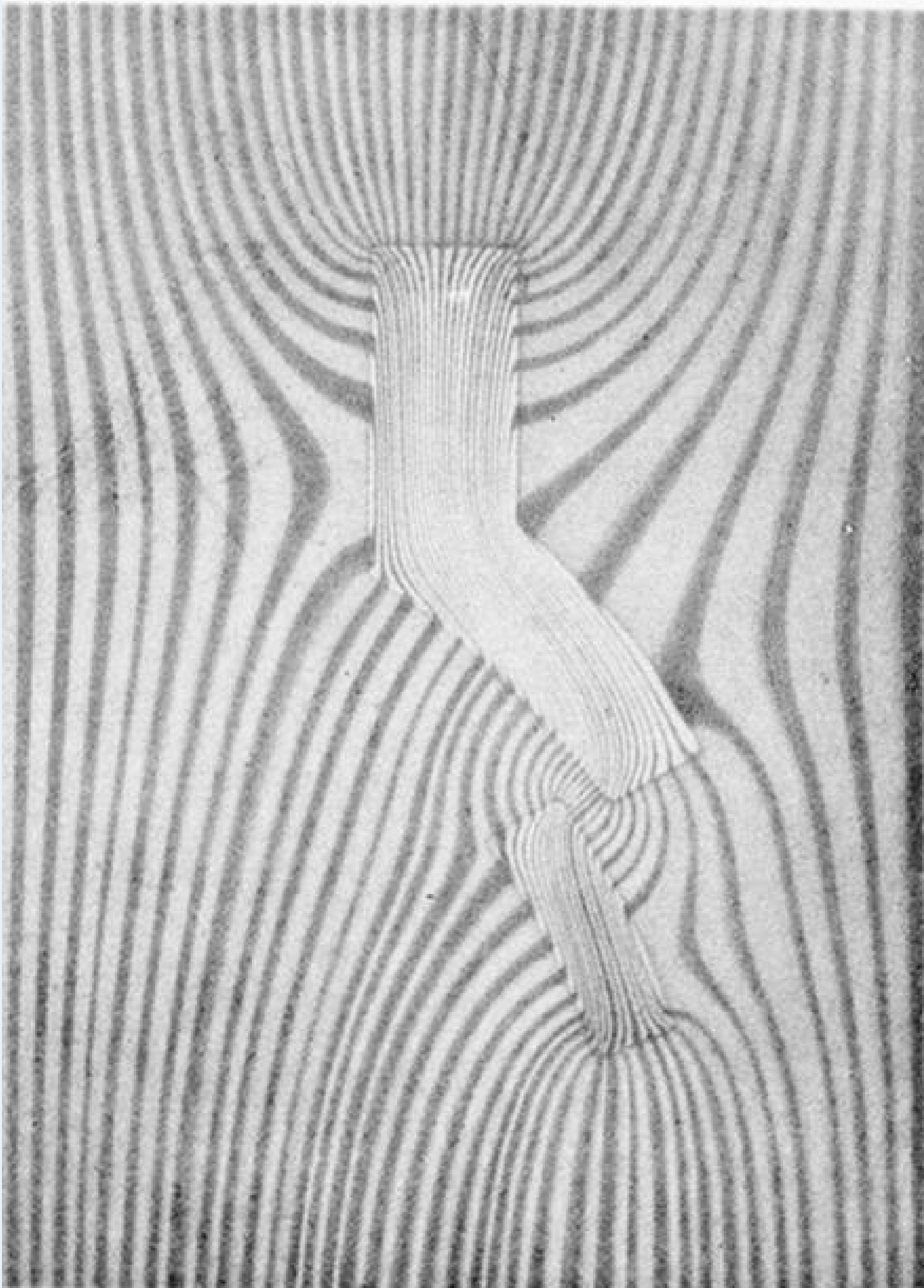

36

ttraction of Armature by Pole-piece.

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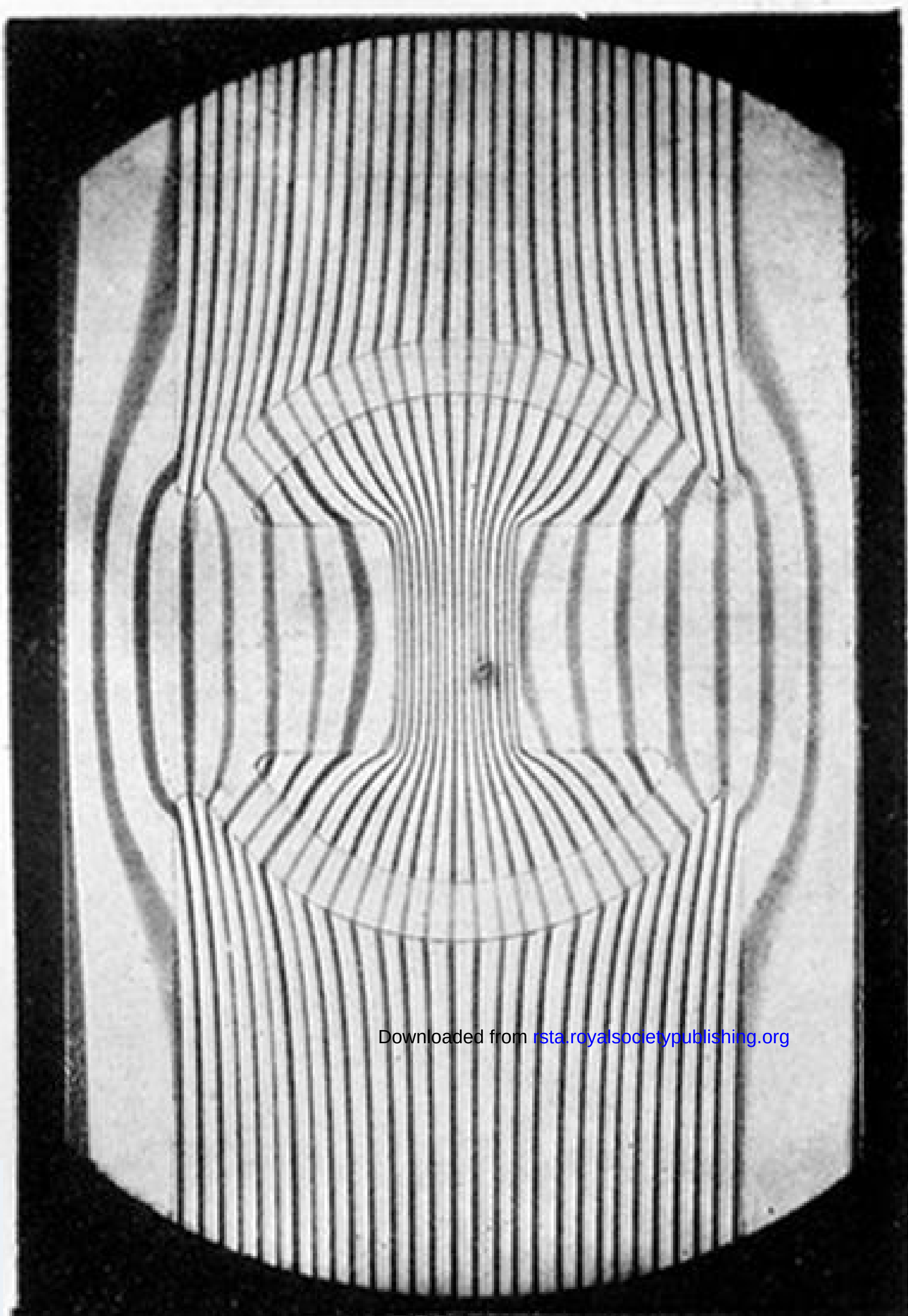
37



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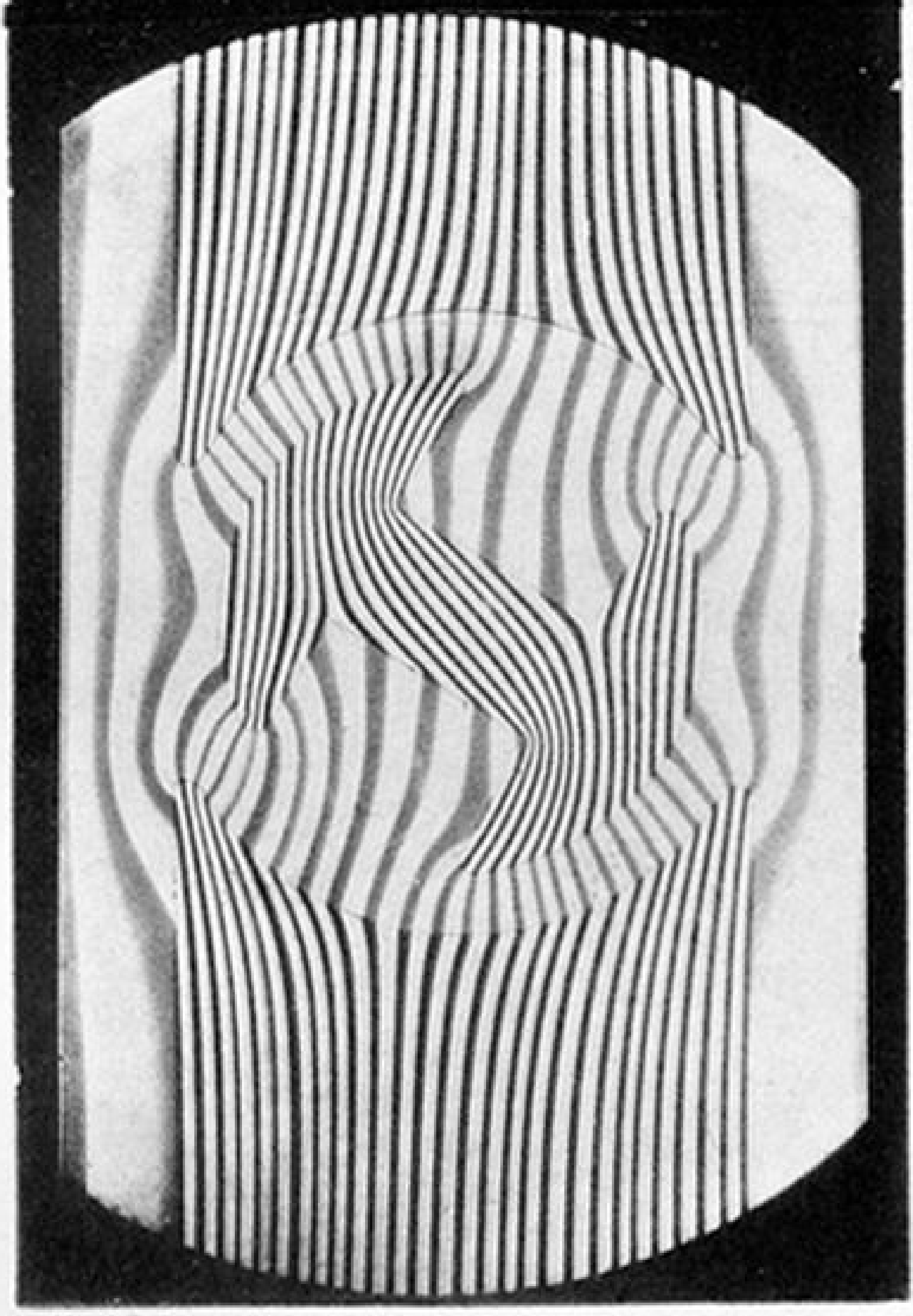
38



Siemens Shuttle-wound Armature in two positions.

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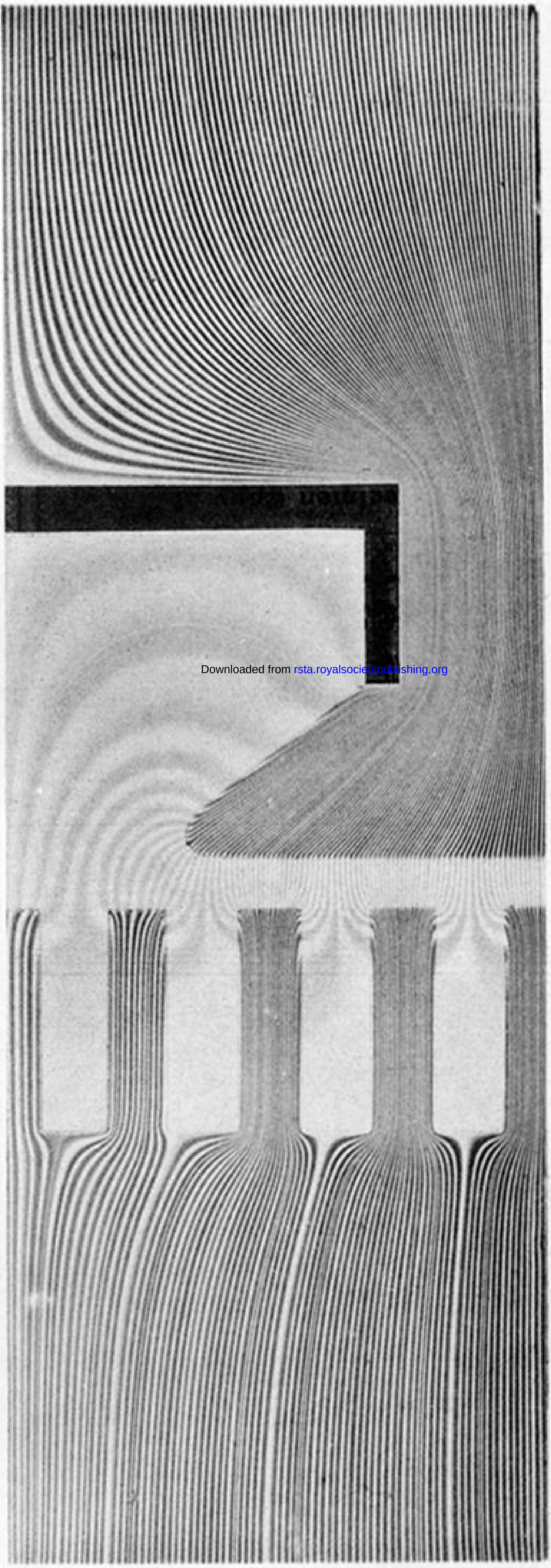

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39

ringe of Magnetic Field near edge of Pole-piece in Dynamo with Toothed-core Armature.

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